Topic Study Group 19

Research and Development in Problem Solving in Mathematics Education

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Aims
This Topic Study Group aims to provide a forum for those who are interested in aspects of problem solving research and development at any educational level, to share recent findings or to exchange ideas. It will also provide an opportunity for the general participants to become acquainted with the progress and current issues of the field, as well as its foreseen future directions. A further goal of the organizing team is that the meetings of this TSG should promote communication and instigate collaborations among the participants. That is, the themes and ideas addressed during the development of the sessions will provide the basis to structure a proposal to write a book on the field.

Themes
Problem solving is the heart of mathematics. The teaching and learning of problem solving has a long history in mathematics education. Problem solving is an activity, which provides students with opportunities to construct and experience the power of mathematics. It is also an instructional approach, which provides a consistent context for students and teachers to learn and apply mathematics.

The primary focus of this TSG is to identify and discuss the current status of research and development in problem solving in mathematics education around the world. More specifically the following areas will be explored:

(1) To understand the complex processes involved in mathematical problem solving;

(2) To explore the process in which students learn and make sense of mathematics via problem solving activities, and how can the teacher facilitate this process;

(3) To discuss ways to evaluate problem solving competencies;

(4) To discuss the role of using computational tools in problem solving approaches;
(5) To identify and discuss future directions of problem-solving research and development.

The program includes round-table discussions, individual presentations, small group presentations and plenary discussions during the development of the sessions. Thus, the sessions are based on discussing the contributions approved by the organizing committee and address issues related to:

(a) **Foundations of problem solving.** Here, we are interested in discussing themes around the principles or tenets that are important in problem solving activities. Questions that can guide the discussion include: What are the main principles or tenets that distinguish a problem solving approach in research and practice? What does it mean to learn mathematics in terms of problem solving activities? What is the role of routine or nonroutine tasks in problem solving approaches? To what extent the practice of development mathematical knowledge is consistent with problem solving approaches? How perspectives like models and modelling or those that emphasize models in general relate to problem solving? To what extent the principles and tenets associated with problem solving have evolved in accordance with the development of computational tools?

(b) **Studies in students' behavior during mathematical problem solving.** These will mostly concern with cognitive, metacognitive, social, and affective aspects of problem solving; accessing knowledge effectively and the interaction in collaborative work are of particular interest. Some relevant questions around these themes involve: How students’ problem solving approaches can be characterized? How have problem-solving approaches evolved in terms of research questions and methods? What are the current trends? What theoretical frameworks have been developed in mathematical problem solving? What is a suitable methodology for studying problem solving processes? How should students’ problem solving competencies be evaluated?

(c) **Instructional approaches: Learning and teaching in problem solving.** What exactly does the student learns from problem solving experiences? Do we wish to be
able to teach mathematical facts or even theories through problem solving? Are present teaching practices effective for all purposes? How to tackle some practical problems, e.g. training teachers, extra time that the problem solving seems to involve.

Questions related to the issues include; what makes a task a "good" problem? How can a problem be used for teaching mathematical topics? To what extent should we expect students to pose and solve their own problems? What types of assessment are consistent with problem solving approaches? To what extent, international assessments like PISA or TIMSS actually evaluate problem-solving competencies?

(d) **Research and development in problem-solving with ICT technology.** The development and availability of computational tools have influenced not only the development of the discipline; but also the way students construct their mathematical knowledge. In this context, we are interested in discussing questions that involve: How can we effectively use technologies (e.g., internet, calculators, computers etc) to facilitate problem solving? How can we effectively use technology to advance problem-solving research? Do computer representations take away some of the initiative from the solver? What types of mathematical reasoning, including mathematical arguments, do students or problem solvers develop as a result of using various computational tools? What types of strategies and problem representations become important in problem solving environments that promote the use of computational tools?

(e) **Curriculum proposals and problem solving.** Some curriculum frameworks (NCTM, 2000) recognize the relevance of problem solving activities. However, there is still a need to discuss ways in which fundamental tenets associated with problem solving need to be organized to support a particular curriculum. How should a curriculum proposal, that enhances a problem solving approach, be organized or structured? What fundamental mathematical ideas and processes should be central in a proposal that promotes problem-solving approaches?
Three Shortcomings of Past Problem Solving Research

1. Relationships are Unclear between Concept Development and the Development of Problem Solving Competencies. One shortcoming of past problem solving research is that it has not been clear how concept development is expected to interact with the development of relevant problem solving heuristics, beliefs, dispositions, or processes. In fact, in most curriculum standards documents (NCTM, 2000; http://www.doe.state.in.us/standards/), problem solving tends to be listed as the name of a chapter-like topic similar to algebra, geometry, or calculus. In other words, the implicit assumption is conveyed that problem solving ability is expected to increase by: (a) first, mastering relevant concepts, (b) second, mastering relevant problem solving heuristics, strategies, beliefs, dispositions, or processes, and (c) third, learning to put these concepts and processes together to solve problems. Consequently, when such assumptions are coupled with the flawed belief that students must first learn concepts and processes as abstractions before they can put them together and use them in “real life” problem solving situations, problem solving tends to end up never getting taught at all in many classrooms. So, one of the most critical challenges for future problem solving research is to clarify the nature of relationships that should exist between concept development and the development of problem solving competencies.
2. Relationships are Unclear between Competencies on Textbook Word Problems and the Levels and Types of Understandings needed to use Mathematics Concepts and Abilities Beyond School. A second shortcoming of past problem solving research is that, because of its almost exclusive emphasis on textbook word problems, mathematics educators have given relatively little attention to the kinds of problem solving competencies that are needed when mathematical thinking is required outside of school. The main exception to this rule has been in the area of elementary arithmetic concepts – where the emphasis has been on skill development rather than problem solving. This neglect of problem solving beyond school is especially significant because, in research about problem solving in fields (such as engineering) that are heavy users of mathematics and technology (Lesh, Hamilton & Kaput, 2007; Zawojewski, Diefes-dux & Bowman, 2008), it has become clear that, in a technology-based age of information, problem solving outside of school tends to be significantly different than problem solving in the context of word problems of the type emphasized in school textbooks and tests (Lesh & Caylor, 2008). Therefore, even if past research on mathematical problem solving would have been successful at explaining students’ behaviors in the context of traditional kinds of textbook word problems, it is unlikely that such explanations would be useful without modification in the kinds of future-oriented problem solving situations that are emerging beyond school. Of specific relevance is that, in the world beyond school, specific questions being answered might not be known until long after the problem (dilemma, decision, discomfort) begins to be addressed. (Knorr-Cetina & Mulkay, 1982; Latour, 1987; Sawyer, 2006)

3. Research on Mathematical Problem Solving has not Accumulated. Failed or flawed concepts or conjectures have continued to be recycled or embellished – with no significant changes being made in the underlying theoretical perspectives. This is exacerbated by the fact that mathematics education researchers have generally avoided tasks that involve developing critical tools for their own use. Unlike their counterparts in more mature sciences (physics, chemistry, biology), where some of the most significant kinds of research often involve the development of tools to reliably observe, document, or measure the most important constructs that are hypothesized to be important, mathematics educators have developed very few tools for observing, documenting, or measuring most of the understandings and abilities that are believed to contribute to problem solving expertise. Furthermore, partly because operational definitions and tools have not been developed to observe, document, and assess the development of most constructs that have been claimed to be important, apparently failed or flawed concepts tend to be continually embellished or recycled. For example, the first paper in this pair describes the continuous embellishment of a theory that focuses on explicitly-learned rules. It also describes instances where “new” theoretical constructs clearly consist of nothing more than new names for previously discredited constructs. The basic problem has been that short lists of rules (e.g., heuristics, beliefs, metacognitive processes) tend to have descriptive but not prescriptive power; longer lists of prescriptive rules become so long that knowing when to use them becomes as important as knowing how to use them. Yet, introducing meta-rules (i.e., higher-level rules that operate on lower order rules) simply transfers the same basic shortcoming to a higher level.

Using new names to recycle old constructs is not the same as producing durable constructs, and continuous embellishment is not the same as theory development. As Popper (1963) emphasized, one of the most important characteristics that distinguishes a scientific theory from an ideology rests on the potential falsifiability of its assumptions and claims. In fact, according to Popper, one of the most important ways that theories develop is by rejecting hypotheses. Yet, rejected
hypotheses rarely occur in the kind of research that has dominated inquiry about mathematical problem solving. So, for mathematics educators who are interested in research on problem solving, what we believe to be most needed are: (a) tools for observing, documenting, and measuring important constructs, (b) theoretical perspectives which do not encourage orthodoxy, and (c) research methodologies which encourage the consideration of diverse ways of thinking – but which also encourage selection among alternatives.

**Shortcomings Associated with Theories & Research Methodologies**

1. **Concerning the development of useful theories.** The theoretical perspective that we will emphasize in this paper traces its roots primarily to Piaget (Piaget and Beth, 1966) and to American Pragmatists such as Pierce, James, Holmes, Meade, and Dewey (Lesh and Doerr, 2003) who were originators of many of the most important ideas underlying modern views of situated cognition and socio-cultural factors that mold and shape knowledge development. Like pragmatism, the theoretical perspective that we emphasize is not so much a theory as it is a framework for developing theories. Like the pragmatists, we believe that no single theory is likely to provide solutions to the complex kind of problems that mathematics educators most need to understand and solve. One reason why this is true is because mathematics education tends to be more like engineering than physics – in the sense that the systems that we need to understand are largely products of human design or human guided development. So, the same conceptual systems that are used to make sense of these products of human development are also used to change them.

How does this challenge the idea that there can be a grand theory of education? Consider the fact that, in fields such as aerospace engineering, we will never have a fixed and final theory of things like space shuttles. One reason why this is true is because as soon as we understand them better, we will change them. So, every conceptual system that is developed for thinking about them is really the \( n^{th} \) in a continuing series. Furthermore, the design of such engineered artifacts usually involves trade-offs among competing perspectives and interests - because we usually want products that are low in cost but high in quality, or low in risk but high in possible gain, and so on. In fields like engineering, such observations have prompted the well known quip that: *Engineering is the science of understanding and designing things when there is not enough time, money, or other resources - and when trade-offs need to be considered which involve conflicting conceptions of success.* Education is much like this – and we will not be able to reach a final grand theory because of it.

In general, in research based on MMP, we adopt the pragmatists’ point of view that what research on problem solving most needs is not another grand theory which claims to explain everything from cooking, to carpentry, to students’ behaviors on textbook word problems (Schoenfeld, 2007). So, we do not expect realistic solutions to realistically complex problems to be solved by single research studies, nor even by single theories. Instead, what are most needed are models which are embodied in artifacts and tools that are designed to be powerful, sharable and reusable. Such models also need to integrate ways of thinking drawn from a variety of practical and theoretical perspectives. Once this is done, model development can lead to theory development; no single theory should be expected to provide guidance for most important problems or decision-making issues related to mathematical problem solving, rather models and theories should guide decision-making.
2. Concerning inadequacies of observational studies and teaching experiments. Two research methodologies which have dominated research of mathematical problem solving are: (a) observational studies in which researchers observe (and often analyze videotapes of) students solving problems, and (b) teaching experiments in which the researcher attempts to validate the importance of some heuristic, belief, disposition, or process by demonstrating that it can be taught.

One shortcoming of observational studies tends to be that constructs that are useful for describing past problem solving behaviors are not necessarily useful for prescribing future problem solving behaviors. A second shortcoming of observational studies is that the things that observers see are always strongly influenced by the researchers’ preconceived notions. This shortcoming is especially powerful during a time when “heavy users” of mathematics claim that beyond school, significant changes have been occurring in the nature of situations where some type of mathematical thinking is needed beyond school.

One shortcoming of teaching experiments is that they simply have not been successful. Small treatments have produced small effects, and large treatments either do not get implemented sufficiently or their complexity tends to makes it impossible to draw causal inferences. So, in either case, such studies usually end up concluding that the researcher didn’t try hard enough – or in the right way. But, beyond these practical considerations, studies designed to show that “it” works confront the same problems that led to the demise of aptitude-treatment interaction theories (ATIs). ATIs assumed that if students can be classified as having one set of profiles or traits as variables, then it should be possible to match the students with suitable treatments, so that particular learning goals will be reliably achieved. However, as Cronbach and Snow (1977) showed, ATI treatments were not effective assessments of student learning because complex interactions amongst the variables often mattered more than the variables themselves; to test all of the combinatorial combinations of the variables was not only computationally impractical task but also theoretically impossible.

Not only did such “treatments” prove to be combinatorically impractical, but they also proved to be theoretically impossible because of feedback loops, second-order effects, and other emergent properties associated with complex systems which have chaotic (unpredictable) outcomes. In fact, many of the most powerful actions that determine success tend to involve two-way interactions (rather than one-way actions) among students, teachers, and other relevant agents or resources (e.g, parents, programs, policy makers, curriculum materials, assessment systems, etc.).

3. Concerning the usefulness of design research methodologies. A recent *Handbook of Design Research in SMET Education* (Kelly, Lesh & Baek, 2008) describe a powerful new class of research methodologies that should be especially useful in research on mathematical problem solving. Although design research methodologies are new to learning scientists and mathematics educators, they have been used for years in many design sciences such as engineering (Zawojewski, Diefes-du, & Bowman, 2008) – where the “thing” that needs to be understood and explained are also being designed by the relevant researchers.

Learning scientists tend to believe that design research methodologies were developed by theoreticians who were attempting to make results of their lab-based research more practical (Brown, 1992) or by software developers or educational program developers who were attempting to provide better theoretical grounding for best practices (Collins, 1992). But, in
mathematics education, and in particular in our own research investigating the nature of students’ developing mathematical concepts and processes, design research methodologies developed mainly out of attempts to minimize the amount of researcher guidance (Lesh, 2002). This work was modeled on Soviet-style teaching experiments (Krutetskii, 1976).

One dilemma that influenced our research was that regardless whether we focus on the development of students’ conceptual systems, or the design of curriculum materials or programs, the “things” that we want to study are “things” that we ourselves are helping to develop or design. So, how can hypotheses be tested when they involve phenomena that are continually changing – and when the changes are partly driven by the conceptual systems that researchers are developing? How can we be certain that principles which are rejected or accepted today will not need to be revisited tomorrow?

A second way to describe the preceding dilemma is to recognize that when we investigate the nature of students’ interpretation abilities, students’ interpretations clearly are influenced by both their own structuring abilities and also by the structure of the tasks that researchers or teachers present. So, when researchers examine students interacting with activities designed by researchers, the interactions and the explanations of the students are not likely to be reducible to simple input-output rules, because researchers are a part of the system they are studying.

To deal with the preceding kinds of issues, it is useful to notice that mathematics educators are not alone! Similar dilemmas confront engineers or other design scientists when they are attempting to understand worlds that they themselves are designing – and worlds filled with feedback loops and second-order effects (where A impacts B, B impacts C, and C impacts A). The following observations about the ways that engineers confront these sorts of issues are useful to consider:

- Regardless whether engineers are designing software or space shuttles, the underlying designs tend to be important parts of the products that are designed. So, when trial products are tested, the underlying design principles also are tested.
- The design “specs” that engineers are given enable them to test products and to choose among strengths and weaknesses associated with alternative designs. This allows engineers to move in directions that are increasingly better without basing decisions on preconceived notions of what is “best.”

On the one hand, such procedures cannot overcome the fact that any current model can only be the $n^{th}$ iteration in a continually evolving series. On the other hand, sequences of such models often provide auditable trails of documentation which reveal important trends or patterns that otherwise would not be apparent. These patterns often enable generalizations to be made.

Models & Modeling Perspectives on Mathematics Problem Solving, Learning & Teaching

Foundations of MMP have been described in a number of recent publications (Lesh & Doerr, 2003; Lesh & Lehrer, 2002; Lesh & English, 2005; Lesh & Sriraman, 2005). MMP evolved primarily out of Piagetian and American Pragmatist perspectives - which also prosed many modern situated and socio-cultural views of problem solving, learning, and teaching (Lesh & Doerr, 2003, p. 519-556). Compared with other theoretical perspectives that have been used to investigate mathematical problem solving: (a) MMP emphasizes interpretation and communication aspects of understanding as much as it emphasizes procedural capabilities, (b)
MMP investigates problem solving processes developmentally – using techniques similar to those that others have used to investigate what it means to understand the development of concepts ranging from early number concepts (Steffe et al., 1983; Clements & Bright, 2003; Fuson, 1992), to rational numbers and proportional reasoning (Lesh, Post & Behr, 1985; Middleton et al., 2001) to the foundations of algebra (Driscoll et al., 2001), statistics (Konold & Lehrer, in press), or calculus (Kaput, 1997), and (c) MMP emphasizes the fact that, as we enter the 21st century, significant changes have been occurring in both the kinds of situations where some type of mathematical thinking is needed for success beyond school, and the levels and types of understandings and abilities that are needed for success in these situations (Lesh, Hamilton & Kaput, 2007). So, even if past theories of problem solving would have proven to be adequate for describing students’ thinking in the context of traditional textbook word problems, MMP research entertains the notion that these theories may need to be modified significantly to describe the kind of mathematical thinking that is needed beyond school in a technology-based age of information (Lesh, Hamilton & Kaput, 2007).

Unlike most past theories that have been used to investigate mathematical problem solving, MMP was not developed primarily to explain problem solving per se. Instead, MMP was designed to investigate the development of mathematics concepts. Nonetheless, MMP also has used developmental studies to investigate what it means to “understand” problem solving processes. These studies have shown that concept development and the development of problem solving processes are closely and synergistically related. For example, one of the earliest questions that MMP researchers investigated was: What is it, beyond the kind of understandings emphasized in most textbooks, tests, and classroom teaching, that enables students to use the things they have learned in real life situations beyond school? (Lesh, Landau, & Hamilton, 1983) Results of these studies made it clear that the following two questions are significantly different.

• What should students do when they are stuck (i.e., when they are not aware of any relevant concepts or processes)?

• What additional levels or types of understanding do students need to develop in order to be able to use concepts and abilities which: (a) are recognized as being relevant to a given problem solving situation, (b) are at intermediate stages of development, and (c) need to be adapted significantly to be useful in current circumstances?

MMP investigates questions such as: (a) What does it mean for students to “understand” relevant heuristics, strategies, beliefs, dispositions, or metacognitive processes? (b) What is the nature of students’ early understandings of relevant heuristics, strategies, beliefs, dispositions, or metacognitive processes? (c) How (or in what ways) do students’ early understandings develop?

In the most general terms, models can be thought of as being systems for describing or designing other systems. As such, they are conceptual systems or interpretation systems, and because they are developed for a purpose, they are purposeful conceptual systems. Then, a distinctive
characteristic of mathematical and scientific models is that they focus on systemic (or emergent\(^1\)) properties of systems-as-a-whole.

One reason why MMP focuses on interpretation abilities is because, outside of school, in virtually every area where researchers have investigated similarities and differences between experts and novices (or between gifted versus average ability students, or between successful versus relatively unsuccessful problem solvers), results have shown that experts not only do things differently, but they also see (or interpret) things differently.

A second reason why MMP focuses on interpretation abilities is because Piaget-inspired researchers have shown that the development of most mathematics concepts depend on the development of students’ abilities to make sense of situations using operational/relational systems-as-a-whole. That is, relevant concepts do not take on their appropriate mathematical meanings until students are able to think systematically. Examples of systemic properties include invariance with respect to a system of operations, transitivity with respect to a system of relations, or properties that involve minimization, optimization, or stabilization of operational-relational systems. So, what Piagetians showed is that, if the conceptual systems that students use to interpret their experiences are not yet functioning as systems-as-a-whole, then students’ thinking tends to be unstable (e.g., they lose the metaphorical “forest” when their attention focuses on “trees” - or vice versa). Their thinking also tends to be characterized by: (a) centering – losing cognizance of one attribute when others are noticed, or (b) conceptual egocentrism – lacking the ability to be self-critical, or to consider alternative ways of thinking.

When we say that modeling is about interpretation and expression, this includes the fact that modeling is about the description and explanation of existing systems, and it also is about the design and development of new systems. According to MMP, the development of mathematical competence is about the development of powerful mathematical models and modeling abilities at least as much as it is about the acquisition of mathematical facts, skills, or processes. Yet, when we focus on the mathematics of description and explanation, this does not mean that the mathematics of computation and derivation are neglected. Such neglect would be as foolish in mathematics as it would be to ignore basic skills in athletics or performing arts – where equal attention also is given to scrimmages, competitions, and performance in other complex decision-making situations where the emphasis is on much more than isolated basic skills – and where knowing when to do things is as important as knowing how to do them.

Because of the preceding perspectives, MMP defines problem solving activities to be goal-oriented activities in which problem solvers need to make significant adaptations to their current ways of thinking in order to achieve the desired goal. Consequently, MMP focuses on problem solving situations in which model development is an important part of the product that problem solvers produce – or the underlying design is an important part of the conceptual tools that are

\(^1\) According to MMP, mathematics is the study of structure (Lesh and English, 2005). If we look at the undefined terms that occur in the formal axiomatic systems that are used to define mathematical concept, then every “undefined term” in these axiomatic systems is an emergent property of the systems. That is, all of its mathematical meaning comes from the systems that are used to define them. Similarly, underlying every statement of value, MMP expects there to be a system of values. Underlying every heuristic or metacognitive process, MMP expects there to be a conceptual system. And, underlying every fact or skill, MMP expects there to be a conceptual system in which the fact or skill becomes meaningful.
designed. Then, when solution development involves conceptual adaptation, at least as much as information processing, it is misleading to characterize problem solving as getting from givens to goals when the path is not obvious. In fact, model development tends to involve several express-test-revise cycles in which significant changes generally need to be made to initial conceptions of givens, goals, and possible solution processes. So, the development of solutions involves the adaptation of existing conceptual systems much more than it involves the search for ideas and procedures which have been misplaced. So, the kinds of heuristics that are most useful are those that help students’ ways of thinking evolve beyond current conceptions – all of which tend to be at intermediate stages of development.

**MMP-based Design Research Studies & Tools to Support Research Collaborations**

According to MMP, researchers, teachers, and students – all are considered to be model developers. Students develop models in response to problems that are simulations of important new kinds of situations where important types of mathematical thinking are needed beyond school in the 21st century. Teachers develop models (and conceptual tools) for making sense of students’ modeling activities. Researchers develop models of interactions between students and teachers. At all three levels, model developers express their current ways of thinking in the form of artifacts or tools which are designed explicitly to be useful for some specifically targeted purpose. Because the “design specs” make it clear that the underlying design (or conceptual system) is an important part of the artifact that is designed, when the artifact or tool is tested the important aspects of the conceptual system that it embodies is also tested. Furthermore, because the artifacts or tools that are produced need to be powerful (in the specific situations in which they were created), sharable (with other people), and reusable (in the future, and in other situations beyond the one in which they were created), they also contribute to community building and to the accumulation of knowledge. So, at all three interacting levels of model development, the products that model developers produce are expected to go beyond simply being tested for usefulness, sharability, and generalizability; they also are designed to have these attributes.

The preceding perspectives lend themselves to multi-tier design studies (Kelly, Lesh & Baek, 2008) which are aimed at investigating interactions among the model development activities of students, teachers, and researchers. Furthermore, multi-tier design studies also are specifically designed to coordinate the work of multiple researchers who are working at multiple sites and who represent a variety of practical or theoretical perspectives which may range from student development, to teacher development, to curriculum development, to theory development. At all levels of multi-tier design studies, many of the most important products that problem solvers produce are powerful, sharable, and reusable tools for their own use. So again, the result is to promote community building and the accumulation of knowledge.

**MMP Alternatives to Past Problem Solving Research Methodologies**

1. **Model-Eliciting Activities - Alternatives (or Supplements) to Clinical Interviews or Videotape Analyses:** MMP research now uses model-eliciting activities in many situations where we once used clinical interviews or videotape analyses – both of which tend to be very labor intensive and difficult to replicate. (Lesh & Lehrer, 2000). Unlike most of the problem solving situations that have been used in research on mathematical problem solving, MEA’s were designed, first and foremost, for research purposes (Lesh & Caylor, 2008). The fact that they also have proven to be useful to support student learning (Lesh & Doerr, 2003), assessment (Lesh &
Lamon, 1992), and teacher development (Schorr & Lesh, 2003) is mostly due to inherent synergies between our views of mathematics education research and practice. Principles for designing MEA’s have been explained in several recent publications (Lesh, et. al. 2000; Hjalmarson & Lesh, 2008).

As their name suggests, MEA’s are activities in which students’ develop a model - or an artifact or a tool which explicitly embodies an important conceptual system (explanation, interpretation, design) that the researcher wants to investigate. Therefore, because the underlying conceptual system is an important part of the designed artifact or tool, testing these products also involves testing the underlying design principles that they embody. Furthermore, because underlying conceptual systems are expressed in forms that can be examined and assessed by students, teachers, and researchers, solutions to MEA’s tend to involve sequences of iterative express-test-revise cycles similar to the kind that are involved in the first-, second-, and n-th-drafts that are involved in the development of other kinds of written or drawn descriptions of situations. Therefore, auditable trails of documentation tend to be produced automatically, and important aspects of the evolving models can be inspected by both students and teachers (or researchers). These documentation trails often supplement the kind of information that can be obtained with time-consuming videotape analyses. Furthermore, because MEA’s are designed so that significant conceptual adaptations occur during relatively brief periods of time (e.g., 60-90 minute problem solving episodes), MEA’s often function something like little Petrie dishes in science laboratories. That is, important developments occur in easily observable forms during sufficiently brief periods of time so that researchers can go beyond observing successive states of knowledge to also observe processes that lead from one state to another. Furthermore, compared with the kind of information that can be gained from clinical interviews or videotape analyses, it often is possible to involve far more students using MEA’s – and the results tend to be far more sharable.

When we compare information that can be gained from MEAs versus clinical interviews, it is noteworthy that a primary goal of clinical interviews is to follow students thinking – rather than simply investigating how close students can come to the researchers preconceived notions about how students should think about important mathematical concepts or processes. But, especially when the kind of thinking that is being investigated focuses on students’ interpretation abilities, every interpretation that students produce is influenced by both the structure of the task and the students’ structuring abilities. Therefore, each time a student develops a new interpretation, interpretation abilities tend to be impacted. And, if the interpretation involves a powerful mathematical construct, these impacts tend to be significant. So, in MMP research, we address this fundamental difficulty by trying to be as explicit as possible about how both students and teachers (or researchers) structure the tasks at hand – and about how they interact.

Whereas, in clinical interviews, researchers adapt their questioning to the thinking revealed by individual students, in MEAs, the students are able to interpret a single problem in a variety of ways and at a variety of levels of sophistication. So, MEAs tend to be self adapting.

2. Local Conceptual Development Studies - Alternatives to Expert-Novice Studies: MMP research often compares problem-solvers-who-are-isolated-individuals to problem-solvers-who-are-groups – in somewhat the same way that other researchers from other theoretical perspectives have compared experts and novices, or gifted problem solvers and average-ability problem solvers. Using MEA’s, one result of this approach is that it has become clear that problem solvers’ early interpretations of model-eliciting activities usually involve a collection of
partly-overlapping yet undifferentiated partial interpretations of different aspects of the relevant situations. So, regardless whether the problem solving is an individual or a group, model development tends to involve gradually sorting out, clarifying, revising, refining, and integrating the preceding kinds of gradually evolving ways of thinking. Furthermore:

(a) The evolution of these initially-unstable communities of constructs tends to resemble the evolution of complex and diverse ecological systems – far more than they resemble the movement of a point along a path (i.e., *getting from given to goals when the path is unclear*).

(b) Heuristics that are intended to help problem solvers make productive adaptations to existing ways of thinking often are significantly different than heuristics that are intended to help problem solvers figure out what to do when they are stuck (with no apparent concepts available).

(c) Heuristics and metacognitive processes evolve in ways that are often quite similar to the dimensions of development that apply to other types of concepts or abilities that mathematics educators have studied. In particular, Vygotsky’s (1978) concept of internalizing external functions often results in early understandings of heuristics that are distinctly social in character. So, instead of “looking at a similar problem” it often is useful to “look at the same problem from another point of view” (and to be aware of the fact that one’s current point of view is not the only possible point of view.

(d) In the context of MEA’s, heuristics and metacognitive processes generally function tacitly rather than as explicitly executed rules; in MEAs, heuristics and metacognitive processes have far less to do with helping students know what to do next, and have far more to do with helping them interpret the situation (including alternative ways of thinking about given, goals, personal competencies, and “where they are” in solution processes). For example, when athletes or performing artists analyze videotapes of their own performances (or those of others), it is useful for them to develop a language for describing these past performances. But, this language usually is not intended to result in prescriptive rules about what to do at some given point in future performances. Instead, the language and imagery that they develop tends to be aimed mainly at helping them make sense of things during future performances. In other words, they are aimed mainly at the development of more powerful models.

3. Multi-Tier Design Research - Alternatives to Japanese Lesson Plan Studies: MMP research often investigates the development of teacher knowledge by engaging teachers in the development of tools for facilitating, documenting, analyzing, or assessing the development of students’ models during MEA’s. For example, one class of teacher-level tools has been referred to as *ways of thinking sheets* (Berry, 2006; Carmona, 2004; Hjalmarson, 2004; Lesh, Doerr, Carmona, & Hjalmarson, 2003). *Ways of thinking sheets* are sharable and reusable tools that teachers often find it useful to develop making sense of students’ work – and for recording alternative ways of thinking that students adopt (a) while they are working on solutions to MEA’s, (b) when they are giving oral reports of their results for MEA’s, or (c) when they submit written reports of their results for MEA’s. For teachers, the purposes of these ways of thinking sheets usually is to help them give students feedback about strengths and weaknesses of their work, or to help them identify appropriate follow-up activities that harvest students’ insights and addresses their needs. Or, for teacher developers, the development of *ways of thinking sheets* often functions similarly to *Japanese Lesson Plan Studies* (Driscoll et al., 2001). Teachers often work in teams of three, and tool development goes through a series of iterative express-test-revise cycles. However, because research has shown that one of the most powerful ways to
positively influence teachers’ teaching practices is to help them become more insightful about the nature of their students’ thinking (Zawojewski, Diefes-dux & Bowman, 2008), and because the development of ways of thinking sheets minimize the amount of time that teachers are taken away from teaching, they lend themselves to effective on-the-job teacher development activities.

- In real classrooms (those that are not ongoing research laboratories), the learning that has occurred through the use of MEA’s has been more impressive when teacher-development and student-development go hand-in-hand.

- In a sense, ways of thinking sheets are MEA’s for teachers. Just like MEA’s for students, they tend to generate auditable trails of documentation about the development of teachers’ knowledge and abilities.

4. Evolving Expert Studies - Alternatives to Ethnographic Observations or Questionnaires: MMP research often involves evolving expert studies in which researchers, and other relevant experts, are engaged in the development of models, artifacts or tools in which the underlying conceptual system is an important part of the product. For example, as discussed in the book, Foundations for the Future in Mathematics Education (Lesh, Hamilton & Kaput, 2007), in the book, Models and modeling in Engineering Education: Designing experiences for all students (Zawojewski, Diefes-dux & Bowman, 2008), and in a series of semester-long follow-up studies with professional engineers, business managers, and others who are heavy users of mathematics, we engaged three-person teams of these experts to work with teachers and learning science researchers to co-design MEA’s which they believed would help clarify insightful and future-oriented responses to the following questions. What is the nature of typical problem-solving situations where elementary-but-powerful mathematical constructs and conceptual systems are needed for success in a technology-based age of information? What kind of “mathematical thinking” is emphasized in these situations? What does it mean to “understand” the most important of these ideas and abilities? How do these competencies develop? What can be done to facilitate development? How can we document and assess the most important (deeper, higher-order, more powerful) achievements that are needed: (i) for informed citizenship, or (ii) for successful participation in wide ranges of professions that are becoming increasingly heavy users of mathematics, science, and technology? How can we identify students who have exceptional potential which are not measured on standardized tests?

Unlike studies in which researchers observe or interview experts, the preceding studies recognized that the opinions of researchers, teachers, and other experts would be certain to evolve significantly if they participated in activities in which they repeatedly expressed their current ways of thinking in forms that were tested and revised iteratively – based on peer review, and based on trials in which their draft activities were tried out with students. In the beginning, these experts’ opinions focused on traditional kinds of skill building, but by the end, there was a consistent and overwhelming consensus that future-oriented problem solving will involve: (a) designing and making sense of complex systems, (b) working in teams of diverse specialists each of whom use continually evolving tools, and (c) participating in multiple-stage projects in which relevant abilities emphasize the multiple-media communicating, collaborating, planning, monitoring, and assessing. Also, computational and multi-media modeling often replaces models that were based on single, solvable, differentiable functions.

References


A TECHNOLOGY-BASED INVESTIGATION OF UNITED STATES HIGH SCHOOL STUDENT MATHEMATICAL PROBLEM SOLVING

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Summary

The purpose of this paper is to discuss an investigation of students’ behavior during mathematical problem solving that may provide new ways for teachers to help students think about mathematics. This study explored 122 high school students’ problem solving strategies, using technology that enabled the researcher to track and model in detail the steps students used to answer mathematical problems. This paper reports findings explaining the different problem solving strategies and metacognitive approaches that students used to solve various types of mathematics items. Based on study findings, the paper also recommends strategies teachers can use to help students become more metacognitively aware of their mathematical thinking. Finally, the paper describes the technology used to document and analyze student problem solving behaviors.

Introduction describing the study and research questions

A major challenge for teachers of mathematics is understanding what students know and what misconceptions deter them from solving problems correctly. Teachers can infer that students who are higher achieving (as measured by grades and test scores) understand more than do students who are lower achieving, but that inference merely allows us to stratify them, not to deeply understand how they are engaging with the mathematics.

The purposes of this study, then, are 1) to identify the strategies that high school students use when solving mathematics problems so as to better understand the processes they use and 2) to uncover some potential reasons females underperform in mathematics compared to males. This study enables a more detailed understanding of student test-taking behavior by providing a more authentic look at what students do before they choose a final answer. Ultimately, determining what enables higher-performing students to respond correctly can inform new ways of conceptualizing instruction.

Theoretical framework

The need to improve students’ mathematical problem solving capacity in the United States can be seen in the performance of U.S. 15-year-olds on the Program for International Student Assessment [PISA] test, in which the U.S. ranked 24th out of 29 developed nations in mathematics literacy and problem solving (Augustine, 2007, OECD, 2004). One reason for the poor performance of U.S. students may be that these students are not provided with instruction that successfully integrates content learning with experiences that foster their understanding of problem solving processes (NRC, 2006).
Problem Solving

To improve mathematics instruction in problem solving, educators must first better understand specifically what it is that students do and understand when they problem solve. Strengthening students’ mathematical problem solving requires that teachers actively engage students in a complex mathematical task in which students construct their understanding of mathematics in nonroutine ways. Performing such a task, however, does not always imply a thorough understanding of that task (Stevens, et al., 2004; 2005; 2006). Students can perform mathematical steps without making the connections necessary to transfer or apply that knowledge in a different context (Lajoie, 2003). Rather than simply performing steps, to solve a problem successfully and understand what they are doing, students need to have knowledge of the content and a strategy to solve the problem. They must monitor their progress through to the solution and must be motivated to tackle the problem and solve it (Marshall, 1995). Instructional approaches that emphasize that students take responsibility for their own learning—that is, approaches that provide them with opportunities to choose, guide, and assess their own task activities and progress—are key to student success in problem solving and in making the connections necessary to apply their problem solving skills in new contexts (Inquiry Synthesis Project, 2006; NRC, 2006).

When approaching mathematics problems, students rely on various resources and types of information (Chi & Glaser, 1985; Ericsson & Simon, 1993; Schoenfeld, 1988). These resources and types of information form the frameworks that students use to interpret and solve different items. Ideally, students are able to identify and interpret a problem sufficiently well to choose the correct framework for solving it, resulting in a correct response. In reality, however, many students sometimes do not know which framework to choose, or choose inappropriate frameworks, which results in inconsistent patterns of correct and incorrect responses (Marshall, 1995; Tatsuoka, 1993). Furthermore, choosing an appropriate framework does not necessarily always lead to a correct answer, because computational errors can also lead to an incorrect response (Tatsuoka, 1990). Thus, trying to explicate students’ problem-solving processes requires that teachers and researchers look beyond their correct and incorrect answers and undertake instead a detailed, empirical investigation of how students organize information.

Every mathematics problem contains a host of concepts that can be linked to a group of specific steps that must be followed to successfully solve the problem. When students choose to follow certain steps in a particular order, they are demonstrating a pathway for organizing information to solve that problem. By assembling detailed information on students’ responses to multiple problems, researchers can trace the steps that students took to solve each problem and evaluate how well their approaches worked on different types of items—at the individual student level and across various groups of students (e.g., by classrooms). Most students refine their problem-solving strategies over time—which is consistent with models of skill acquisition (Ericsson, 2004)—gradually using fewer steps and eventually settling on a preferred approach (Stevens, Soller, Cooper, & Sprang, 2004; Stevens & Soller, 2005). Researchers can analyze student self-regulation and self-monitoring of strategies by investigating these steps, which will provide a better understanding of the complexity of student problem-solving performance (Hartley & Bendixen, 2001; Song & Hill, 2007).
IMMEX: Using Technology to Study Problem Solving

Given the myriad ways that students can organize information and regulate the steps they take to solve problems, the project of truly understanding what students are thinking and doing as they work problems can seem insurmountable. Advances in technology, however, have provided one way to get inside students’ heads, as it now enables us to track in detail the actual steps students take to solve problems and the time they spend on each step (Hartley & Bendixen, 2001).

One pioneering technology that enables better understandings of student thinking is the Integrated (now Interactive) Multi-Media Exercises (IMMEX) program, which draws on both case-based (Schank, 1990) and production system (Newell & Simon, 1972) models of problem solving. Such a tool gives a more qualitative look at how students solve problems, since it captures in intricate detail the variety of approaches they can take. This captured information opens the door to a deeper understanding of the comprehensive nature of student thinking because rather than analyzing only a student’s final answer, this tool allows researchers to look at each step that led to that answer.

IMMEX presents a problem to be solved in a problem space—that is, a space with a finite set of concepts, numbers, and equations that students must combine in order to create a solution path. Within the IMMEX problem space, various drop-down menus provide pathways the students can choose from. A problem space typically will not encompass all the combinations that students could use to solve problems, but it does provide an essential preliminary view for better estimating what they are able to do (Tatsuoka, 1990, 1995). Further, although IMMEX’s simulated problem spaces are finite, they do provide enough different types of information that students with diverse math backgrounds could successfully solve the problems.

Working in IMMEX, students can assess the problem structure—the information needed to solve the problem—and then organize a mathematical representation—an arrangement of that information into a series of steps that solves the problem (Bennett, Morely, & Quardt, 1998). Most students want to arrive at an answer and will follow some process to produce one. If their chosen process leads to a wrong answer, they will probably try a different process if given a chance to try again. IMMEX software allows researchers to track all the steps—forward, backward, and sideways—that students take as they attempt to solve problems. IMMEX also records and displays the sequence of steps and the time spent on each one (Ericsson & Simon, 1993; Stevens, 1991).

Design and Methodology

This study analyzed—and compared across gender—the problem-solving strategies of 122 high school students in their junior and senior years (72 females and 50 males) on a set of SAT mathematics problems administered both on paper and in IMMEX. I also interviewed a subset of students for their feedback and impressions about solving problems online, a mode that forced them to show their steps.
Methods

To understand the specific steps that students take when tackling different kinds of mathematical problems, this study used Interactive Multi-Media Exercises (IMMEX; www.immex.ucla.edu), which consists of a library of online multimedia simulations for problem solving. IMMEX has a refined set of modeling tools for monitoring student’s performance and progress (Stevens et al., 2004, Stevens & Soller, 2005; Soller & Stevens, 2007). I presented students with 25 mathematics items on the IMMEX platform. For each problem, IMMEX provided students with various menus to choose from, in a simulated problem space that is composed of a finite but representative set of concepts, numbers, and equations that students can combine to create a solution path. With these menus, students had to identify, define, and represent their steps to solve the problem. They developed reasons for choosing information that might or might not be productive in helping them find an answer (Baxter & Glaser, 1997; Stevens, et al., 2003; 2004; 2005; 2006). This case-based paradigm simulates situations with sufficient information that students with diverse experiences can successfully synthesize and thus solve the problem. IMMEX allows for detailed investigation of the steps and procedures that students use to complete a task, because all steps are documented by sequence and time spent per step.

I analyzed students’ strategies to see which students planned and self-regulated their learning. I found that the more successful problem solvers tended to take time initially to identify the constraints of the problem and the steps necessary to solve it before embarking on those steps. As a result, they took fewer steps, used fewer nonproductive steps, and looked ahead by assembling their procedures before acting on them. By breaking a problem into manageable and familiar steps, successful problem solvers can chunk information into concepts and hierarchies that facilitate good problem solving (Baxter & Glaser, 1997; Paek, 2002; Siegler, 1988). This finding suggests that teachers should help students become more purposeful in (or regulate) their problem solving strategies.

Procedure

Participants first completed a retired SAT-I test from the 10 Real SATs (College Board, 1997) under standard SAT time constraints using paper and pencil. Next, students used IMMEX to solve 31 SAT-I mathematics problems that came from the same SAT-I test the students had completed for the first part of the study. Eight students also participated in a focus group to reflect on their problem solving strategies and to explain their thinking behind their steps.

Developing the IMMEX Problem Space

I created a specific problem space in IMMEX for solving each of the 31 mathematics items along with a method for using and coding the search path maps. The problem space included formulas, definitions of concepts, and a breakdown of the process for arriving at a solution. The problem space for this study was developed based on a formal task analysis the researcher had already conducted, for which students listed the steps they used to solve certain math items. Additionally, the researcher incorporated into the problem space common errors that students make in arithmetic, basic algebra, and geometry (Tatsuoka, 1990, 1995). These two
elements (the formal task analysis and the incorporation of common errors) helped to determine the menus and submenus needed for the IMMEX platform in this study so that the majority of students could solve the problems using the information given (Mislevy, Yamamoto, & Anacker, 1991).

The problem space was the same for each of the 31 items in IMMEX, except for some of the submenus, which were changed to correspond with the proper substeps and the numbers and equations related to each problem. The problem space included the math concepts necessary for a correct solution as well as bugs and distractors, which were included to track where students made arithmetic errors or had misconceptions about the problem. Students could easily navigate through all the menus and submenus and still not be able to correctly solve a problem—to reach an accurate solution, they needed to know what kinds of information were pertinent and be able to order that information correctly. Within the IMMEX problem space, each problem was presented with five main menus. The problem and the five main menus were always at the top of the screen, even when students navigated through the submenus. Each main menu represented one of five math concepts: arithmetic, angles, area, perimeter, and solving equations. Clicking on one of these menus revealed a host of submenus also representing math concepts. Clicking on a submenu led to a series of equations and/or numbers. These equations/numbers were represented in expanded form, so that the student had to decide where to combine or collapse terms. The menu structure included shortcuts so that students could collapse several steps into one. For example, consider the equation $2x + 10 = 5 – 3x$. The traditional way of solving it would be first to move the numbers to one side by subtracting 10: $2x + 10 – 10 = 5 – 10 – 3x$. To get the variables on one side of the equal sign, the next move would be to add $3x$ to both sides: $2x + 3x = –5 – 3x + 3x$. Then both sides of the equation would be divided by 5: $5x/(-5) = -5/5$ to arrive at the final response of $x = -1$. In IMMEX, the menu shortcuts enabled students to collapse these three steps, computing the information in their heads so they could move to the answer in one step.

The IMMEX problem space also included common arithmetic errors students could make that were associated with the distractors offered on the paper-and-pencil test. In the problem above, for example, students could incorrectly subtract by $3x$ so that the response would be $2x – 3x = -5$, resulting in a final answer of $x = 1$. These types of mistakes were included as options in the submenus and the equation structures. The purpose of these incorrect paths was to document where students might go wrong in coming up with their final answer choice.

After completing all the work and arriving at an answer, students entered their responses after clicking the “solve problem” button. The study was structured to give students two chances to solve the problem so that they could reconsider each step they had taken and so that the researcher would have an opportunity see how students revised their steps to answer the problem correctly on the second try. The second chance also allowed students who might have made a simple arithmetic error to backtrack and correct it.
Research results

Number of Steps Taken

Using the data IMMEX collected, the number and types of steps the students took to solve each problem were analyzed. In general, the fewer steps a student had taken in attempting to solve a problem, the more likely it was that the student had solved the problem correctly—as students who unsuccessfully completed a problem tended to take more irrelevant steps that were not helpful to solving the problem. In addition, the number of steps taken differed between males and females. On average, males took two fewer steps than females did to solve a problem: males averaged four or fewer steps (M = 3.91, SD = 2.06), whereas females averaged six or more steps (M = 6.12, SD = 1.77). Even with these differences, females attempted more IMMEX items, took longer to solve each problem, and answered more problems correctly than males did; the reason for the higher success of females on these items appears to be that they verified their steps, not that they were inefficiently taking extra steps. On the paper-and-pencil SAT-I mathematics sections, however, females averaged lower than did the males. The IMMEX test had no time constraints, so it may be that females performed better in the untimed situation than males did.

Informal interviews with the participants suggested that the females liked to be sure of their answers and would use any available information to verify them. They wanted their answers to be correct on the first attempt, and they took more steps and more time to ensure correctness. The interviews suggested that males, on the other hand, tended to be inclined toward an answer and would select it, knowing they had a second chance if they got it wrong. This method resulted in fewer steps and less time taken per problem. Males indicated they were also more likely to guess once they had eliminated some choices.

Amount of Time Spent per Step and per Problem

I also analyzed time spent per step and per problem. Females tended to take 2 s more per step (M = 19, SD = 21) than did males (M = 17, SD = 18), which resulted in females taking about 55 s more per problem (M = 2:06, SD = 2:51) than males (M = 1:11, SD = 1:45). These differences are statistically significant (p < 0.01). This difference in time spent per step, coupled with the fact that females took more steps than males did, may well help to explain why females tend to score lower on standardized mathematics tests: They are not able to complete as many problems.

Number of Attempts Made

Students were given two opportunities to solve each IMMEX problem, so I could document the changes they made in their steps from the first to the second attempt. The majority (61%) of students solved the problems correctly on the first try, and an additional 23% answered the problems correctly on their second try. The steps students took on the second tries for both correct and incorrect answers were analyzed. Students who correctly solved a problem on the second try demonstrated an orderly process in which they deliberately retraced their steps to verify their answers and more systematically regulated their steps, indicating that these students
knew what they were doing but had made a small error in computation at some point. Students who did not solve the problem correctly on the second try, on the other hand, showed less organization and planning in their process.

Observing the processes used by the participants gives an inside view of how they regulated their learning as they solved problems, and suggests reasons for the performance differences documented between females and males. The amount of time and number of steps to solve each problem varied between males and females, with females taking extra steps to verify their answers and therefore taking more overall steps per problem than males. This verification process resulted in females spending more time on each problem than did males. The extra steps and time, however, paid off in the females performing slightly higher than the males on the IMMEX problems.

**Discussion**

The results confirmed the outcomes from previous research (e.g., Gallagher, 1990; 1992; Gallagher & De Lisi, 1994) about differences in gender in problem solving: females tended more than males to follow algorithms and verify their steps and the answer before moving on to the next problem. In the interviews, females articulated a greater need for verification in their work.

This research shows the importance of understanding in detail the steps that students take when solving mathematics problems. Tracing students’ steps allows researchers to better understand how students organize information when coming up with an answer. In the present study, tracing students’ steps illuminated some of the reasons that females’ math test scores are typically lower than males’ scores. Finding out what knowledge students possess no longer needs to be surmised only from final answers and scores, as the use of IMMEX in this study demonstrates. Researchers can and must continue to probe the processes that students employ and the knowledge they bring to bear when confronted with a mathematics problem. The more deeply that educators and researchers can analyze student thinking, the better we can measure students’ competence, knowledge, and abilities—and thus the better we can design tools and practices for teachers to teach them effectively.

This study contributes to the literature on the mechanics of students’ problem solving (e.g., Baxter & Glaser, 1997; Lajoie, 2003; Marshall, 2005; Stevens, et al., 2003; 2004; 2005; 2006). The findings from this study also show how technology such as IMMEX can provide two key strategies for improving mathematics teaching and learning: it can provide teachers with access to student thinking that is usually not obvious and thus enable them to modify instruction appropriately (Pellegrino, et al. 2001), and it can provide opportunities for students to reflect and fine-tune their problem solving strategies, giving them a strong context for thinking about and being successful in mathematics. A main implication of this study is for educators to find ways to increase students’ metacognitive skills in mathematics so that when they participate in assessments, their performance reflects their actual understanding rather than their habitual approaches to problem solving.

The differences that the present study found in self-regulation of problem solving demonstrate the importance of teaching students how to plan their problem-solving. An
underutilized strategy in instruction is having students practice using an overarching schema for new and novel problems. Other strategies include having students plan their steps before they actually begin to solve a problem, and then having them reflect on these plans and actions. In fact, a five-stage problem-solving process that is recommended in most textbooks and resources emphasizes that students should begin by reading and understanding the statement of the problem, then analyze the given facts, and then propose steps to solve the problem. These three stages are the planning phases, which should take place before students actually work the problem. The final two stages are carrying out the planned steps and then verifying the solution. On the basis of this study, a sixth stage is recommended: Teachers should provide opportunities for students to reflect on the steps they took to solve the problem. This allows students time to reflect on their previous work and, if necessary, plan a better set of steps for upcoming problems.

References


Formulating mathematical conjectures in learning activities, assisted with technology

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Summary

What types of activities should professional development programs include to revise and extend high school teachers’ mathematical and pedagogical knowledge? We propose a route to engage high school teachers in an inquiry approach to reflect on their current practice and to construct hypothetical learning trajectories that can eventually guide or orient the development of their lessons. In this report, we focus on the activities that were worked within a professional community that include the participation of mathematicians, mathematics educators and doctoral students.

Introduction

What mathematical and pedagogical knowledge should the education of high school mathematics teachers include? Who should participate in the educational programs to prepare mathematics teachers? What should be the role of mathematics departments or the faculty of education in preparing prospective and practicing teachers? What types of educational programs should practicing teachers participate in order to revise and extend their mathematical knowledge and to incorporate research results from mathematics education into their practices? Traditional ways to prepare high school teachers normally involve the participation of both mathematics departments and the faculty of education. Mathematics departments offer courses in mathematics while the faculty of education provides the didactical or pedagogical courses. This model of preparing teachers has not rendered solid basis to help teachers provide an instructional environment in which they exhibit mathematical sophistication to interpret and prompt students’ responses and to organize and implement meaningful learning activities for their students. Indeed, it is common to read that university instructors complain that their first year university students lack not only fundamental mathematical knowledge; but also strategies or resources to solve problems that require more than the use of rules or formulae.

Many practicing teachers, for different reasons, have not learned some of the content they are now required to teach, or they have not learned it in ways that enable them to teach what is now required. … Teachers need support if the goal of mathematical proficiency for all is to be reached. The demands this makes on teacher educators and the enterprise of teacher education are substantial, and often under-appreciated (Adler, et al., 2005, p. 361).

Davis and Simmt (2006) suggest that teachers’ preparation programs should focus more on teachers’ construction of mathematical ideas or relations to appreciate their connections, interpretations, and the use of various types of arguments to validate and support those relations, rather than the study of formal mathematics courses. Thus, the context to build up their mathematical knowledge should be related to the needs associated with their instructional practices. “… [mathematical knowledge] needed for teaching is not a watered version of formal mathematics, but a serious and demanding area of mathematical work” (Davis and Simmt, 2006, p. 295). In this work, we report that teachers’ mathematical knowledge can be revised and enhanced within an interacting intellectual community that fosters an inquisitive approach to
develop mathematical ideas and to promote problem-solving activities. The core of this community should include mathematicians, mathematics educators, and practicing teachers. This community promotes collaborative work to construct potential learning trajectories to guide or orient the teachers’ instructional practices. Teachers need to be interacting within a community that supports and provides them with collegial input and the opportunity to share and discuss their ideas in order to enrich their mathematical knowledge and problem solving strategies. In this context, we illustrate the importance of using computational tools to represent and explore various ways of approaching mathematical tasks.

**Research questions**

Several research works (Santos-Trigo, 2004; Schoenfeld, 1994, 2000; NCTM, 2000) emphasize the importance of formulating and validating conjectures when learning and developing mathematics. Conjecturing processes involve several dimensions when technology is used systematically. For instance, the idea of generalizing is widely amplified and at the same time one has the opportunity to ask questions about the way in which a particular computational system works. The questions that guided this research are: What type of mathematical reasoning might be developed by high school teachers in order to reconstruct or enhance their mathematical knowledge when using technology to explore hypothetical learning trajectories? What type of mathematical arguments might high school teachers use to explain unexpected computer mathematical results?

**Conceptual Framework**

The conceptual framework is structured around two main theoretical issues: (i) problem solving and technology and (ii) hypothetical learning trajectories. We have chosen these constructs since learning mathematics is achieved through problem solving, which is enhanced by using technological tools. In this regard, we argue that promoting an inquiring approach when learning mathematics can be attained effectively by formulating questions and elaborating conjectures systematically. This path is strongly related with the finding and exploration of different hypothetical learning trajectories.

**Problem solving and the use of technology**

In problem solving activities it has been recognized the relevance of an inquiring or inquisitive approach for teachers to work on mathematical tasks or to think of and reflect on their instructional activities. In this context, learning mathematics or developing mathematical knowledge through problem solving is conceptualized as working with tasks where:

A task, or goal-directed activity, becomes a problem (or problematic) when the “problem solver” (which may be a collaborating group of specialists) needs to develop a more productive way of thinking about the given situation (Lesh and Zawojewski, 2007, p. 782).

It is important to clarify what is understood by a productive way of thinking. According with the same authors “…Developing a ‘productive way of thinking’ means that the problem solver needs to engage in a process of interpreting the situations, which in mathematics means modeling” (p. 782).

In this context, an important component is to develop an inquiring way of thinking to formulate questions, to identify and investigate dilemmas, to search for evidences or information, to discuss
solutions, and to present or communicate results. This means willingness to wonder, to pose and examine questions, and to develop mathematical understanding within a community that values both collaboration and constant reflection. At this point Schoenfeld (1994) argues: “Mathematicians develop much of that deep mathematical understanding by virtue of apprenticeship in to that community [mathematical community]–typically in graduate school and as young professionals” (p. 68). A mode of inquiry involves necessarily the challenges of the status quo and a continuous re-conceptualization of what is learned and how knowledge is constructed.

[In a community of inquiry] participants grow into and contribute to continual reconstitution of the community through critical reflection; inquiry is developed as one of the forms of practice within the community and individual identity develops through reflective inquiry (Jaworski, 2006, p. 202).

Taking this view into account, and considering that the use of technology has been playing an important role in the process of mathematical learning by enhancing different elements of mathematical thinking, particularly formulating and validating conjectures, it is relevant to ask: what is the role of a computer system in the process of posing and justifying conjectures? How trustable are the results obtained with the aid of a computer system?

Concerning the first question Santos (2007) argues: “A relevant aspect when representing a task with the aid of a dynamical software is that students have the opportunity to pose questions about the structure of some elements of the configuration” (p. 124).

Regarding the second question, Dick (2007) has introduced the term Mathematical Fidelity “to emphasize that the mathematics of the tool does not always represent the mathematics as it is understood by the mathematics community” (p.1174). In the example that we will discuss, it will be pointed out the strong necessity of providing mathematical arguments to deal with discrepancies between the computers results and the expected ones.

**Hypothetical Learning Trajectories**

To promote the teachers’ inquiring approach to their practice we rely on the construction of Hypothetical Learning Trajectories (HLT). These trajectories emerge from examining potential routes of solution of mathematical tasks. Simon and Tzur (2004) state that the construction of hypothetical learning trajectories is based on the following assumptions:

1. Generation of an HLT is based on understanding of the current knowledge of the students involved.
2. An HLT is a vehicle for planning learning of particular mathematical concepts.
3. Mathematical tasks provide tools for promoting learning of particular mathematical concepts and are, therefore, a key part of the instructional process.
4. Because of the hypothetical and inherently uncertain nature of this process, the teacher is regularly involved in modifying every aspect of the HLT (p. 93).

In this perspective, we suggest that teachers together with other community members (including mathematicians), work on various ways to approach the tasks and to identify relevant concepts and problem solving strategies needed to solve them. We argue that this type of teachers’ interaction becomes relevant to foster and to develop a problem solving approach that involves:

- Seeing the mathematical content in mathematically unsophisticated questions, seeing underlying similarity of structure in apparently different problems, facility in drawing on different mathematical representations of a
problem, communicating mathematics meaningfully to diverse audiences, facility in selecting and using appropriate modes of analysis ("mental", paper and pencil, or technological), and willingness to keep learning new material and techniques (Cohen, 2001, p. 896).

In addition, we also recognize that the use of computational tools offers to teachers the opportunity to enhance relevant aspects of mathematical thinking as well as to represent and examine mathematical tasks in terms of questions that can lead them to develop or reconstruct some mathematical results. For instance, the use of a dynamic software allows teachers to represent problems dynamically in order to recognize and explore mathematical relations within a geometrical configuration, and to identify loci described by members of the configuration when others are moved. In this context, the use of computational tools becomes important for teachers to discuss pedagogical paths associated with the hypothetical learning trajectories that can be useful to guide or orient their instructional practices.

We claim that the inquiring process is strongly intertwined with the appearance of hypothetical learning trajectories derived from a problem solving activity. By this we mean that in the process of formulating questions, there arises the opportunity to learn or reconstruct new mathematical concepts that emerge while pursuing those questions.

**Research Design, Methods and General Procedures**

The Center for Research and Applied Mathematics, that is part of a public university, is in charge of developing and implementing a professional program to revise and improve high school teachers’ mathematical and didactical knowledge. As a part of the program, we coordinated a 40 hrs instructional module, out of four, whose main aim was to illustrate and discuss the strengths and limitations of using computational tools in problem solving activities. To this end, a group that includes two mathematicians, one mathematics educator, and two doctoral students met together during two months, in sessions of three hours a week, to select and discuss the tasks that later would be used during the development or implementation of the activities with the teachers. During each session, one member of the group presented one problem and provided information related to its relevance or rationale and ideas about possible solutions. Then, all participants became engaged in the solution process and at the end they discussed and summarized the main ideas and ways used to approach the task. In this report, we focus on presenting general features of hypothetical learning trajectories that emerged as a result of working on those tasks that later we used to structure the development of the sessions with the actual group of teachers who participated in this module. Our unit of analysis is the work done and reported by the participants (the mathematicians, the math educator and the doctoral students) as a group. Thus, in presenting the results we use the word group to identify and characterize that work. The selected problems came from textbooks and research articles. There were also those that include the construction of an initial dynamic configuration (using dynamic software) in which teachers could formulate their own questions or problems.

**The task:** This problem involves an extension of a task discussed in Santos, et al. (2006, p. 125). In particular, the working group constructed a hypothetical route for teachers to develop an inquiring approach to the tasks in which the use of technology is encouraged. The problem arises from analyzing invariance and structure of simple components of a geometric configuration in order to identify an instructional path to foster the teachers’ construction of mathematical relations.
Given a straight line \( L \), a point \( P \) in \( L \) and a point \( Q \) not in \( L \), draw the segment \( PQ \), a line \( L_1 \) perpendicular to \( PQ \) through \( Q \) and a line \( L_2 \) perpendicular to \( L \) through \( P \). Call \( R \) the intersection of \( L_1 \) and \( L_2 \). What is the locus of \( R \) when \( P \) runs on \( L \)?

![Figure 1: What is the locus of point R when point P is moved along line L?](image)

**Research Results and Discussion**

Figure 2 shows the different stages that appeared during the solution process of the task as well as main features that guide the construction of the different hypothetical learning trajectories. It should be mentioned that the diagram shows only one instructional route that the group proposed during the discussion of the task, however there were two more possible routes (finding the triangle with minimum area and the case where the locus is a hyperbola), whose discussion is not presented.

In this perspective, the meaning associated with the main stages that characterize the potential instructional trajectory involves: (i) the recognition of the high school teachers’ knowledge base to represent and explore the initial task, (ii) the recognition that the aim of the developed task is to provide conditions in order that high school teachers reinforce and reconstruct their mathematical concepts in such way that this would help them to design and guide learning activities in the classroom, (iii) the discussed problem arises from analyzing minimal elements in a geometric configuration with the objective of designing learning tasks and (iv) the possibility that the teachers will bring into the discussion additional elements to modify every aspect of the hypothetical learning trajectory after they have solved the task.

One of the members of the discussion group suggested to approach the problem using Cabri-Geometry to construct the geometric configuration, after this, using the tool *Locus*, it was asked the software to describe the locus drawn by point \( R \) when \( P \) moves on \( L \). Cabri-Geometry shows a graph that looks like a parabola, Figure 1. With this information, some of the members of the group went further in conjecturing, using Cabri-Geometry's tool *Equation or coordinate*: the equation of the locus described by \( R \) corresponds to a parabola. At this point there was consensus that formal arguments were needed in order to continuous with the analysis to find connections and generalizations.

**Using a coordinate system.** An algebraic approach becomes important to construct an argument to show that the locus is a parabola. Here, the group used a Cartesian System in a proper position to facilitate algebraic operations.

Without loss of generality, one can assume that \( L \) coincides with the \( x \) axes, \( P = (t,0) \) and \( Q = (a,b) \). In order to determine the coordinates of \( R \), one finds the equations of \( L_1 \), which turns
out to be $y - b = \left( \frac{(t - a)}{b} \right) (x - a)$, the equation of $L_2$ is $x = t$. Solving the system determined by these two equations yields $y - b = \left( \frac{x - a}{2} \right)^2 / b$… (*), which is in fact the equation of a parabola since $a$ and $b$ are fixed.

**Figure 2:** Hypothetical learning process in the context of a particular task.

At this stage, the dynamic representation of the task becomes a departure point to identify and explore diverse mathematical relations. Here, we document ways in which the working group explored the following general cases:

(a) Same assumptions on $L$, $P$ and $Q$ but now, it was taken an additional point $Q'$ on the segment
$PQ$ and the line $L_1$ that passes through the point $Q'$. What is the locus described by $R$ when $P$ moves along $L$? How does the locus change when $Q'$ moves along the segment $PQ$?

An interesting part of the use of Cabri Geometry to formulate conjectures is that after proving the result the discussion group obtain more accurate information about the parabola. For example, knowing the focus and the directrix, the group could formulate the result in terms of synthetic geometry.

Let $L$ be a line, $Q$ a point not in $L$, $P \in L$, $L_1$ the line that passes through $Q$ and $P$. Take a point $Q' \in L_1$ and draw the perpendicular line to $L_1$ that passes through $Q'$, calling it $L_2$. Through $P$, $Q'$ and $Q$ draw perpendicular lines to $L$, calling these lines $L_3$, $L_4$ and $L_5$, respectively. Let $T$, $S$ and $R$ be the points of intersection of the lines $L$ and $L_4$; $L$ and $L_5$; $L_2$ and $L_3$, respectively. Through $Q'$ draw a perpendicular line to $L_4$ that intersects $L_3$ and $L_5$ at $E$ and $V$ respectively. Let $F$ and $W$ be points on $L_5$ such that $WV = VF = QS^2/4Q'T$. Let $L_6$ be the perpendicular to $L_5$ that passes through $W$ and intersects $L_3$ at $U$. Then $L_6$ and $F$ are the directrix and focus of a parabola with vertex at $V$.

Proof. The claim is equivalent to show that $UR=FR$. We have:

$$FR^2 = VE^2 + (UR - 2VF)^2 \ldots (1)$$

From the similar triangles $PQ'T$ and $PQS$ one has:

$$\frac{QS}{QT} = \frac{VE}{Q'E},$$

and from this, one obtains:

$$VE = \frac{SO}{TQ} Q'E.$$

![Figure 3: We have to prove that $UR = FR$.](image)

Substituting the value of $VF$ and $VE$ in equation (1) and developing the binomial one arrives to:

$$FR^2 = \frac{SO^2 Q'E^2}{QT^2} + UR^2 - UR \frac{QS^2}{QT} + \frac{QS^4}{4QT^2}$$

$$= UR^2 + \frac{SO^2}{QT} \left( \frac{Q'E^2}{QT} - UR + VF \right).$$

From the triangle $RQ'P$ we have $Q'E^2 = (PE)(ER)$; on the other hand $PE = Q'T$, hence from the previous equation one concludes that:

$$FR^2 = UR^2 + \frac{SO^2}{QT} \left( ER - UR - VF \right).$$

We also have $ER - UR = EU = -VW = -VF$; from which the conclusion follows.

In the last result, the group assumed that the second coordinate of the point $Q'$ does not change; with this in mind some of the members of the group ask a very natural question. What would
happen if this condition is replaced by: the distance from $Q$ to $Q'$ remains constant?

(b) Assuming that $L, P$ and $Q$ are as above, but now the point $Q'$ is the intersection of the line $L'$, passing through $P$ and $Q$, and the circle $C$ of radius $r$ with center at $Q$. The lines $L_1$ and $L_2$ are constructed as before, and so is $R$. What is the locus described by $R$ when $P$ moves along $L$? How does the locus behave when $r$ approaches zero?

![Figure 4: What is the locus of point $R$ when point $P$ moves along line $L$?](image4)

![Figure 5: What is the locus of point $R$ when point $P$ moves along line $L$?](image5)

In discussing part (b), with the use of Cabri Geometry the group has the chance to experiment and observe the behavior of the locus generated by $R$. One first approach shows results as shown in Figure 5, and it seems that the locus is a parabola, the *Equation tool* from Cabri Geometry even suggests that we are dealing with a parabola.

Nevertheless, taking a closer look at the geometrical behavior of the locus generated by $R$, there appears a graph as the one shown in Figure 6, which cannot be identified with the graph of a parabola. With this evidence, it is natural to ask for formal arguments to find out which kind of geometric object is described by point $R$. After performing calculations using a Coordinate System the group found that:

$$R = \left( x, \frac{1}{b} \left[ (x-a)^2 + b^2 \pm r \sqrt{(x-a)^2 + b^2} \right] \right),$$

where the center of the circle is $(a,b)$. It should be noticed that the second coordinate of $R$ approaches $\left[ (x-a)^2 / b \right] + b$ when $r$ approaches zero, which is the same result as (*), page 6. This result is consistent with the process of generalizing, an important aspect of the mathematical thinking.

Also the participants asked questions related with the way that Cabri performs geometric transformations. This led to think about the reliability of mathematical results obtained with the aid of a computer system. Here the group had the opportunity to point out the necessity of
analyzing the process and results obtain from the technological tool and ask questions related to the axiomatic system of it.

Closing remarks

Mathematical tasks are key elements of any professional development program that aims to revise and enhance teachers’ mathematical and didactical knowledge. How should those tasks be discussed with teachers in order to identify explicitly ways of reasoning that are consistent with mathematical practice? We argue that tasks or problems should be addressed openly within an inquisitive community that promotes collaboration and mathematical reflection. In this process, the use of computational tools becomes relevant to represent some tasks dynamically and visualize diverse mathematical relations embedded in those tasks. It is evident that the conceptualization of the task as dilemmas, provide the opportunity to identify and explore relations, to open diverse lines of thinking or reflection that can lead the community or the problem solver to approach the task from diverse angles or perspectives. For example, the visual and empirical approach becomes important to identify relevant information, possible relations, and plausibility of solutions. The use of dynamic software offers the opportunity of utilizing particular heuristic strategies (searching for partial solutions) to solve the problem. Thinking of various approaches to the problem, another relevant problem solving activity, allows the problem solver to identify fundamental properties of the solution and possible relations or connections. Thus, problem solving is a continuous activity in which contents (from various domains), resources and strategies are used to initially construct a hypothetical learning trajectory that can be useful to orient and structure the practice of mathematical teachers. Finally, the group that worked on the task recognizes the relevance of approaching them within an inquisitive or inquiring community. The participants have developed a guide to implement the tasks. Of course, the plan and activities to implement the tasks in the professional development program were based on considering the trajectories that emerged during the group sessions.

An aspect, which is of crucial importance when using technological tools for solving mathematical problems, is related to providing support or formal arguments to results produced through the use of the tools. It is well accepted that technology is a powerful tool, however the results obtained should be examined rigorously in order to be accepted or rejected. In this respect Dick (2007, p. 1175) has introduced the term mathematical fidelity and has identify three areas in which a lack of mathematical fidelity can emerge: (i) mathematical syntax, (ii) underspecifications in mathematical structures and (iii) limitations in representing continuous phenomena with discrete structures and finite precision numerical computation. However these areas might not consider aspects related with reliability such as the results in the discussed example. We think that results that disagree with the expected ones has to do with the internal processing of the tool; related with this we suggest that a closer examination of the mathematical structure of the tool has to be done. At this respect our opinion agrees with Zbiek et al. (2007) whose statement is:

As technology becomes an increasingly integrated part of school mathematics, careful analysis of issues of mathematical fidelity [and reliability] will be needed. This type of research will necessitate intense collaboration involving mathematicians, computer scientist, and mathematics education researchers (p. 1176).

We also consider that this analysis should include categorizing levels of reliability of the tool. For instance we claim that the basic arithmetic operations (addition and multiplication within the precision range of calculators and computers) are 100% reliable. This is not the case for more sophisticated mathematical operations.
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References

PROBLEM POsing PERFORMANCE OF Grade 9 students in Singapore on an open-ended stimulus

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Abstract

This is an exploratory study into the individual problem posing characteristics of 152 Grade 9 students (aged 15) from four secondary schools in Singapore. The subjects were novice problem posers in that they were not given any training in problem posing skills. Each student was asked to write down a problem for their friends with the final answer as 60°. Students also solved their own problems. The relationship between the structures of the posed problems, the topics involved in the problems and the solutions were discussed. Students’ self-reported metacognitive regulatory strategies, the effects of achievement levels and of gender were also discussed. It was found that direct proposition type of problems occurred in about half of the posed problems. The presence of problem over-conditioning was not significant across achievement levels and gender. Students’ confidence in their posed problems were found to be related to some of the metacognitive strategies at the property noticing phase, problem construction stage and during solution checking.

Keywords: Mathematical Problem Posing, Problem Solving, Angle Measure, Metacognition

The importance of problem posing in relation to mathematical explorations has been highlighted in various literatures. There are studies linking mathematical problem posing to creativity (Silver, 1994, Haylock, 1987) and to mathematical competence (Ellerton, 1986). Specifically the relationship between mathematical problem posing performance and problem solving abilities has also been studied in recent years. In fact, English (1997) noted that both problem posing and problem solving are closely related and that the process of problem posing in fact draws heavily on the processes of problem solving.

Silver and Cai (1996) made a distinction between two notions of problem posing. Firstly, problem posing can be construed to be a case of a generation of new problems from a mathematical situation. Secondly, it can also be interpreted as the reformulation of a given problem in which there is an intention to uncover the deeper underlying structures of a given question or problem. In this case, it is one strategy in problem solving where the solver tries to answer related questions which will give insights to the original problem. For the present work, the focus will be on the nature of problem posing itself and not as part of a problem solving heuristic. Problem posing is used here as the formulation of new problems from a mathematical stimulus.

The inclusion of activities in which students generate their own problems had also been strongly endorsed by the National Council of Teachers of Mathematics (1991). It is believed that such activities can provide a glimpse of students’ understanding of mathematical concepts and processes and their attitudes towards problem solving. As part of the metacognitive aspect of the national mathematics curriculum framework, problem posing is also strongly encouraged in the
classroom. In the Ministry of Education document, *Mathematics Syllabus (Lower Secondary)* (CPDD, 2001), students are encouraged to “create, formulate or extend problems.” (p.16)

One strand of the studies in problem posing involves developing problem posing as an instructional intervention to improve problem solving skills and to improve disposition towards solving. Some of these include work done by Gonzales (1994, 1998) on using problem posing to improve on preservice teacher training and Manouchehri (2001), who worked on an instructional model for promoting problem posing in a sixth grade classroom. Another strand of work goes into analyzing the problems posed in terms of their surface structures. For example, Marshall’s schema theory (Marshall, 1995) was used in the study by Charalambous, Kyriakides and Philippou (2003) on the problem posing skills of primary school students. For both strands, the contexts of these studies mainly involved mathematical word problems and largely on arithmetic. Subjects of these studies varied between students of various grade levels to undergraduates in pre-service teacher preparation courses.

Lesser work is being done in the area on the cognitive processes of mathematical problem posing itself and the regulation of these processes. Christou, Mousoulides, Pittalis, Pantazi and Sriraman (2005) had proposed in their study on 143 Grade 6 students in Cyprus, a few processes that can be used to describe problem posing. Selecting quantitative information is one of the processes involved in posing problems. It is mostly linked to tasks that require students to pose problems that are appropriate to specific given answers. Such a process involves the ability to focus on the context of the problem structure and the relationship between the given initial information and the subsequent information that the students created to make the posed problems coherent. This is an important skill in building connections across domains of knowledge and sense making in mathematical exploration. One purpose of this study is to look at the selecting process involved in problem posing in the area of school geometry in Singapore. Students’ posed problems in this area will shed light into how they perceive linkages between the different topics within school geometry and into how they construct their problems. The control of this cognitive process is also an important aspect in the study of problem posing.

Livingston (2003) referred to metacognitive regulatory processes as those that one uses to control cognitive activities and more importantly to see to the meeting of the cognitive goal. These involve planning, monitoring of cognitive activities and checking of outcomes of those activities. The other purpose of this study is to illuminate some of these regulatory processes that are involved in problem posing.

**Method**

**a) Subjects**

This is an exploratory study about the individual problem posing characteristics of 152 Grade 9 (aged 15) students from four secondary schools in Singapore. The subjects were novice problem posers. Besides their classroom experience in asking questions, they were not given any specific training in problem posing prior to this study. The decision to locate the study with these students was that few such studies had been made in this area in Singapore.

**b) Task**

Each student was asked to freely write down a problem for his or her friends to solve with the final answer as 60°. Students also solved the problems they had posed. By solving their own problems, the students can make explicit the selecting process as they construct the problem structures. Silver (1990) in her study on problem posing involving number sense, also noted that
such open ended stimulus tends to provide good opportunities for students to be engaged in generative aspects of mathematical thinking.

Much of what constitutes a problem is dependent on the context in which the problem is posed. When students posed problems to friends, they do so with some perceived knowledge of their friends’ familiarity of topics and methods, cognizance of common errors made by friends and the time taken for their friends to complete the tasks. Such perceived knowledge is captured in students’ posed problems.

c) Questionnaire

Immediately after completing the task, students were asked to complete an 18-item questionnaire as shown in Table 1. The purpose of the questionnaire is to get a snapshot of their metacognitive regulatory strategies during their posing and solving. Each item has a 4 point Likert scale with 1 being strongly disagree and 4 being strongly agree. This instrument is an adaptation of Goos, Galbraith and Renshaw (2000) metacognitive survey for secondary students in the Australian state of Queensland in their study on the metacognitive aspects of students solving combinatorics problems. In order to make the questionnaire more appropriate for the students in this study, some of the questions were changed. Further modifications had also been made to take into account the different phases of metacognitive regulatory behaviour in problem posing. These phases were the results of an earlier work made by the authors as they worked on the think-aloud

Table 1

<table>
<thead>
<tr>
<th>Code</th>
<th>Statements</th>
<th>Phase</th>
</tr>
</thead>
<tbody>
<tr>
<td>4A</td>
<td>I read carefully when there is an important information</td>
<td>PN</td>
</tr>
<tr>
<td>6A</td>
<td>I am good at recalling what my teacher had taught</td>
<td>PN</td>
</tr>
<tr>
<td>8A</td>
<td>I ask myself questions about the information before I begin</td>
<td>PN</td>
</tr>
<tr>
<td>9</td>
<td>I use my own examples to make what is given more meaningful</td>
<td>PN</td>
</tr>
<tr>
<td>13A</td>
<td>I try to use my own words when I read the new information</td>
<td>PN</td>
</tr>
<tr>
<td>1A</td>
<td>I check periodically if I am getting the problem that I want</td>
<td>PC</td>
</tr>
<tr>
<td>2AA</td>
<td>I consider other possibilities to a problem before I ask it</td>
<td>PC</td>
</tr>
<tr>
<td>10AA</td>
<td>I find myself checking on my understanding as I posed the problem</td>
<td>PC</td>
</tr>
<tr>
<td>11A</td>
<td>I draw diagrams to help me understand while posing the problem</td>
<td>PC</td>
</tr>
<tr>
<td>17A</td>
<td>I think about the method of solution first before I pose the problem</td>
<td>PC</td>
</tr>
<tr>
<td>5A</td>
<td>I ask if I have considered all possibilities to my problem while solving it</td>
<td>CS</td>
</tr>
<tr>
<td>14A</td>
<td>I go through over new information that is not clear</td>
<td>CS</td>
</tr>
<tr>
<td>15AA</td>
<td>I constantly look back at the problem as I start the solution</td>
<td>CS</td>
</tr>
<tr>
<td>16A</td>
<td>I check my solution as I worked on it</td>
<td>CS</td>
</tr>
<tr>
<td>3A</td>
<td>I know how well I have done once I finish the problem</td>
<td>LB</td>
</tr>
<tr>
<td>7A</td>
<td>I ask if there was an easier way to pose after I finish posing the problem</td>
<td>LB</td>
</tr>
<tr>
<td>12A</td>
<td>I ask if I could have posed a different problem after I finished</td>
<td>LB</td>
</tr>
<tr>
<td>18A</td>
<td>I like the problem that I posed</td>
<td>LB</td>
</tr>
</tbody>
</table>

PN: Property Noticing, PC: Problem Construction, CS: Checking Solution, LB: Looking Back
protocols of 10 students prior to this study. Students were then engaged in the same task of creating problems with the final answer as $60^0$. Property noticing describes the initial phase before students start the active construction of their problems. In this phase, students make associations with the topics that first come to their mind when confronted with this stimulus. Within the problem construction phase, students draw upon their earlier experiences about topics to come up with problems. Simultaneously they are also checking the solutions to their posed problems and retrospectively going back to their earlier posed problems (and modifying when necessary) and see if their solutions make sense. In the last phase, students reflect back and evaluate their work.

One limitation to this study is that the sample of 152 students can not meant to be representative of the students in all the secondary schools in Singapore. The sample size also does not allow for factor analysis of the metacognitive regulatory strategies.

Results

The results of analysis of students’ posed problems, their solutions and the questionnaire responses are presented in two parts. In the first part, problems posed are described in terms of the types of problem structures, the domains of knowledge used and the solutions to the problems. Secondly, discussions are made on students’ self-reported metacognitive regulatory strategies. Specifically the relationship between the different strategies used and the types of posed problems, students’ gender and students’ achievement levels are also discussed. All results are discussed at 5% level of significance.

a) Characteristics of Posed Problems

i) Problem structure

There are direct proposition problems where the solutions require single-step solutions as shown in Figure 1. Each of the direct proposition problems involves a single topic. These problems account for 50.7% of the total posed problems. Their solutions involved some forms imitative reasoning. For example, the solution may involve the recalling of a simple algorithm like finding the angle sum in a triangle as in Figure 1 or a possible memorized answer like $\cos^{-1}(0.5) = 60^0$ to solve “what is $x$ if $\cos x = 0.5$?” These may suggest that students just created them from what first came into their mind without trying to create linkages with other topics. The other possibility is that these are problems which students perceived their friends are able to solve. They also reflect what can be
commonly found as exercises in school textbooks or perhaps problems which they commonly encountered in their classroom learning experiences.

*Figure 2*
Example of a multiple topic problem

![Diagram of a multiple topic problem](image)

The rest of the problems involve multiple steps and are situated in a combination of topics. One example is shown in Figure 2. These questions are good examples of how students were able to link topics together in their problem construction. In the problem in Figure 2, the student involved the uses of the geometric properties of a parallelogram, a trapezium and a triangle. Justifications of the steps were also made by using of the properties of alternate angles and the angle sum of a triangle.

Within these non-direct propositions, there are problems which are over-conditioned. An over-conditioned problem contains extraneous information which does not contribute to the solution. In Figure 3, angle 40° is not needed as part of the solution. This occurs in

*Figure 3*
Example of an over-conditioned problem

![Diagram of an over-conditioned problem](image)

20.4% of the total posed problems. Yet there are other problems that contain inconsistencies in their structures. The angle sum of a triangle is violated in Figure 4.
The problem was constructed without considering the linkage to the other parts of the diagram. The student perhaps was trying to impress his or her friends the sophistication the problem by including more information. Of the posed problems, 17.1% of them are inconsistent problems.

**Figure 4**
Example of an inconsistent problem

### ii) Domains of Knowledge

The problem in Figure 1 involves a single topic about the sum of angles in a right angle triangle. For single topic problems, students also made use of the topics on alternate angles, complementary/supplementary angles, corresponding angles and trigonometric ratios. Multiple topics, namely, the use of the properties of angles in a triangle and alternate angles are found in the problem in Figure 2. Across the four schools, students used a variety of topics in school geometry in their posed problems. Of these topics, the use of angle sum in a triangle is most prevalent. It is observed in 43.4% of the work with others using the topics about complementary/supplementary angles (24.3%), circle properties (21.1%) and alternate angles (15.1%).

As shown in Table 2, the use of the topic on triangle is also prevalent among problems that involve multiple topics (64.2%) rather than in single topic problems (27.1%). Perhaps students are more familiar with this topic. Over-conditioning is also not significantly associated to problems with multiple topics, $\chi^2(1) = 0.588$, $p = .366$.

**Table 2**
Analysis of posed problems involving multiple topics

<table>
<thead>
<tr>
<th></th>
<th>Multiple Topics (P10)</th>
<th>$\chi^2(1)$</th>
<th>Asymp Sig. (2 sided)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Absent</td>
<td>Present</td>
<td>Total</td>
</tr>
<tr>
<td>Use of Angle Sum in Triangle (K3)</td>
<td>Absent</td>
<td>62 (72.9%)</td>
<td>24 (35.8%)</td>
</tr>
<tr>
<td></td>
<td>Present</td>
<td>23 (27.1%)</td>
<td>43 (64.2%)</td>
</tr>
<tr>
<td>Use of Circle (K5)</td>
<td>Absent</td>
<td>78 (91.8%)</td>
<td>42 (62.7%)</td>
</tr>
<tr>
<td></td>
<td>Present</td>
<td>7 (8.2%)</td>
<td>25 (37.3%)</td>
</tr>
</tbody>
</table>

### iii) Solution

There are varying numbers of steps in students’ solutions to their posed problems. They range from the single step direct proposition type to more involved types like the problem in Figure 2. In the course of working through their solutions, students demonstrated they
were able to justify their steps appropriately. Most of the solutions involve the recall of solution algorithms in school geometry and trigonometry. Except for four solutions that have computational errors, the rest of the students’ solutions are found to be correct.

b) Achievement and Gender

Across the four schools, students’ scores in the standardized national Primary School Leaving Examination at Grade 6 are used to classify the achievement levels. Students are classified either as High Achievers (HA), Average Achievers (AV) or Low Achievers (LA). The distribution of gender and achievement levels is shown in Table 3.

Table 3
Distribution of Gender and Achievement Levels

<table>
<thead>
<tr>
<th></th>
<th>Male</th>
<th>Female</th>
<th>Subtotal</th>
</tr>
</thead>
<tbody>
<tr>
<td>LA</td>
<td>11</td>
<td>29</td>
<td>40</td>
</tr>
<tr>
<td>AV</td>
<td>31</td>
<td>41</td>
<td>72</td>
</tr>
<tr>
<td>HA</td>
<td>17</td>
<td>23</td>
<td>40</td>
</tr>
<tr>
<td>Subtotal</td>
<td>59</td>
<td>93</td>
<td>152</td>
</tr>
</tbody>
</table>

The students’ achievement levels are found to be not significantly associated to the types of posed problems, domains of knowledge and their solutions. For example, there is no strong evidence to suggest that HA students produce more multiple topics type of questions compared to students in the other levels. In Table 4, achievement levels and the presence of over-conditioned problems are also found not to be significantly related in this study. This suggests that the problem posing performance of students to the given open-ended stimulus is not strongly influenced by how well they had performed in their standardized tests.

Across the achievement levels, there are also no significant associations with the students’ metacognitive regulatory strategies except for the presence of checking during the solution phase. There is some association between HA students and their self-declared use of checking in their solutions. Lower number of HA students checked their solutions compared to the other two groups of students. This perhaps reflects the HA students’ confidence in their solutions to their posed problems and hence the lesser need for checking.

Table 4
Achievement Profiles, Over-Conditioning and Use of Checking

<table>
<thead>
<tr>
<th></th>
<th>Achievement Profiles</th>
<th>( \chi^2(2) )</th>
<th>Asymp Sig. (2 sided)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>LA (77.5%)</td>
<td>31</td>
<td>36 (90.0%)</td>
</tr>
<tr>
<td></td>
<td>AV (75.0%)</td>
<td>54</td>
<td>43 (59.7%)</td>
</tr>
<tr>
<td></td>
<td>HA (90.0%)</td>
<td>36</td>
<td>29 (72.5%)</td>
</tr>
<tr>
<td>LA</td>
<td>31 (77.5%)</td>
<td>54 (75.0%)</td>
<td>36 (90.0%)</td>
</tr>
<tr>
<td>AV</td>
<td>9 (22.5%)</td>
<td>18 (25.0%)</td>
<td>4 (10.0%)</td>
</tr>
<tr>
<td>HA</td>
<td>31 (77.5%)</td>
<td>54 (75.0%)</td>
<td>36 (90.0%)</td>
</tr>
<tr>
<td>Use of Checking in Solution</td>
<td>No (42.5%)</td>
<td>17</td>
<td>43 (59.7%)</td>
</tr>
<tr>
<td></td>
<td>Yes (57.5%)</td>
<td>23</td>
<td>29 (40.3%)</td>
</tr>
</tbody>
</table>
| Problem structures are also found not to be significant in discussing gender. Like in achievement levels, the over-conditioning feature in problems is also found not to be significant across gender. Table 5 shows some significant results from the questionnaire survey and gender. At property noticing phase, more females than males agree that they were not good at recalling what the
teachers had taught. For both gender, close to half of the students (46.7%) felt that they were good at recalling what had been taught.

More males reported that they had considered other possibilities to the posed problems as they were solving the problems. But most of them (69.5%) did not check their solutions. More males than females are also found to like the problems they had posed compared to females. For all the students, 88 (57.9%) reported they like their posed problems.

Table 5
Questionnaire Responses and Gender

<table>
<thead>
<tr>
<th></th>
<th>Gender</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Female</td>
<td>Male</td>
<td>Total</td>
<td>χ²(1)</td>
</tr>
<tr>
<td>Good at recall (PN) (Q6A)</td>
<td>56 (60.2%)</td>
<td>26 (44.1%)</td>
<td>82</td>
<td>3.788</td>
</tr>
<tr>
<td></td>
<td>37 (39.8%)</td>
<td>33 (55.9%)</td>
<td>70</td>
<td></td>
</tr>
<tr>
<td>Think of method of solution first (PC) (Q17A)</td>
<td>46 (49.5%)</td>
<td>18 (30.5%)</td>
<td>64</td>
<td>5.320</td>
</tr>
<tr>
<td></td>
<td>47 (50.5%)</td>
<td>41 (69.5%)</td>
<td>88</td>
<td></td>
</tr>
<tr>
<td>Consider all possibilities to problem (CS) (Q5A)</td>
<td>45 (48.4%)</td>
<td>18 (30.5%)</td>
<td>63</td>
<td>4.755</td>
</tr>
<tr>
<td></td>
<td>48 (51.6%)</td>
<td>41 (69.5%)</td>
<td>89</td>
<td></td>
</tr>
<tr>
<td>Use of Checking in Solution (CS) (Q16A)</td>
<td>48 (51.6%)</td>
<td>41 (69.5%)</td>
<td>89</td>
<td>4.76</td>
</tr>
<tr>
<td></td>
<td>45 (48.4%)</td>
<td>18 (30.5%)</td>
<td>63</td>
<td></td>
</tr>
<tr>
<td>Like posed questions (LB) (Q18A)</td>
<td>46 (49.5%)</td>
<td>18 (30.5%)</td>
<td>64</td>
<td>5.32</td>
</tr>
<tr>
<td></td>
<td>47 (50.5%)</td>
<td>41 (69.5%)</td>
<td>88</td>
<td></td>
</tr>
</tbody>
</table>

From Table 6, among students who reflected that they knew how well they had done once they had finished posing their problems, 87.8% of them checked their solutions and 86.7% drew diagrams to help them understand as they constructed their problems. Similarly, of those who felt that they had done well, a high number also reported that they asked questions about the information during the property noticing phase of their problem posing. The strategy of asking questions at the property noticing phase, drawing diagrams during their problem construction phase and checking of their solutions appears to account for the higher confidence in their knowing of how well they have done in their problem posing.

Table 6
Questionnaire Responses with Knowing How Well When Done

|                                    | Know how well when done (Q3A) |          |          |       |
|                                    | No                       | Yes      | Total    | χ²(1) | Asymp Sig. (2 sided) |
| Ask questions about information (PN) (Q8A) | No                       | 30 (55.6%) | 34 (34.7%) | 64     | 6.216 | .010                  |
|                                    | Yes                      | 24 (44.4%) | 64 (65.3%) | 88     |       |                        |
| Draw diagrams to help understand (PC) (Q11A) | No                       | 20 (37.0%) | 13 (13.3%) | 33     | 11.575 | .001                  |
|                                    | Yes                      | 34 (63.0%) | 85 (86.7%) | 119    |       |                        |
| Use of Checking in Solution (CS) (Q16A) | No                       | 31 (57.4%) | 12 (12.2%) | 43     | 35.005 | .000                  |
|                                    | Yes                      | 23 (42.6%) | 86 (87.8%) | 109    |       |                        |
Conclusion and Implications

To encourage a variety of problem structures, the classroom teacher needs to broaden the types of problem experiences being presented to students. The teacher can capitalize on the informal activities situated in students’ daily activities and get students to the habit of recognizing mathematical situations wherever they might be and making connections to various aspects of school geometry. Otherwise, students would only be comfortable with constructing direct proposition type of problems which does not allow them to explore the inter-connectedness of topics. This ability to make connections is an important skill in mathematical exploration.

Perhaps, to get students to have more confidence in their problem posing, the teacher can encourage students to ask more questions about the given stimulus during the property noticing phase, to draw diagrams during the problem construction phase and to check their solutions. These metacognitive strategies appear to help novice problem posers in this study to have more confidence in their work.

Since achievement levels and gender are not significant across problem posing performance, classroom problem posing activities should be encouraged for all students. The teacher can also make use of students’ problem posing work as teaching points. For example, the teacher in discussing students’ posed problems, can sensitize the class to issues about the inconsistencies in problem structures or to the notion of over-conditioning in problem construction. Perhaps such discussions may help to produce better problem posers and may contribute to students’ engagement in more quality mathematical inquiry in the classroom.

The teacher in teaching is also involved in posing problems. The school curriculum planner can look into ways of promoting the teacher’s competency in problem posing just like the way problem solving heuristics are made known to teachers. The very way in which the teacher asks questions can affect that shared spirit of investigation between the teacher and the students. Appropriate use of varied problem types which may depart from the textbook exercises may bring about a better quality of classroom interaction. But such teacher’s behaviour is also dependent on the teacher’s beliefs and perceptions about problem posing itself and about the teachers’ views on the nature of mathematics. This is an issue that warrants further investigation.

References


ABSTRACT
Since the 1960s, numerous studies on problem solving have revealed the complexity of the domain and the difficulty in translating research findings into practice. The literature suggests that the impact of problem solving research on the mathematics curriculum has been limited. Furthermore, our accumulation of knowledge on the teaching of problem solving is lagging. In this first discussion paper we initially present a sketch of 50 years of research on mathematical problem solving. We then consider some factors that have held back problem solving research over the past decades and offer some directions for how we might advance the field. We stress the urgent need to take into account the nature of problem solving in various arenas of today’s world and to accordingly modernize our perspectives on the teaching and learning of problem solving and of mathematical content through problem solving. Substantive theory development is also long overdue—we show how new perspectives on the development of problem solving expertise can contribute to theory development in guiding the design of worthwhile learning activities. In particular, we explore a models and modeling perspective as an alternative to existing views on problem solving.

INTRODUCTION
Research on mathematical problem solving has received a good deal of attention in past decades. Among the notable developments have been Polya’s (1945) seminal work on how to solve problems, studies on expert problem solvers (e.g., Anderson, Boyle, & Reiser, 1985), research on teaching problem solving strategies, and heuristics and fostering metacognitive processes (e.g., Charles & Silver, 1988; Lester, Garofalo, & Kroll, 1989), and, more recently, studies on mathematical modeling (e.g., Lesh, in press; English, 2007). Existing, long-standing perspectives on problem solving have treated it as an isolated topic, where problem solving abilities are assumed to develop through initial learning of concepts and procedures followed by practice on “story problems,” then through exposure to a range of strategies (e.g., “draw a diagram,” “guess and check”), and finally, through experiences in applying these competencies to solving “novel” or “non-routine problems.” As we discuss later, when taught in this way, problem solving is seen as independent of, and isolated from, the development of core mathematical ideas, understandings, and processes. Despite these decades of research and associated curriculum development, it seems that students’ problem solving abilities still require substantial improvement especially given the rapidly changing nature of today’s world (Kuehner & Mauch, 2006; Lesh & Zawojewski, 2007; Lester & Kehle, 2003).

This current state of affairs has not been helped by the noticeable decline in the amount of problem solving research that has been conducted in the past decade. A number of factors have been identified as contributing to this decline. These include the discouraging cyclic trends in educational policy and practices, limited research on concept development and
problem solving, insufficient knowledge of students’ problem solving beyond the classroom, the changing nature of the types of problem solving and mathematical thinking needed beyond school, and the lack of accumulation of problem solving research (Lesh & Zawojewski, 2007). Before considering each of these contributing factors, we offer an overview of research on mathematical problem solving over the past 50 years.

A BRIEF SKETCH OF FIFTY YEARS OF RESEARCH ON MATHEMATICAL PROBLEM SOLVING

In mathematics education, research on problem solving has focused primarily on word problems of the type emphasized in school textbooks or tests – where “problems” are characterized as activities that involve getting from givens to goals when the path is not obvious. With such situations in mind, Polya’s book How to Solve It (1945) introduced the notion of heuristics – such as draw a picture, work backwards, look for a similar problem, or identify the givens and goals (later referred to as strategies by mathematics educators) – which mathematics education researchers immediately recognized to be useful for generating after-the-fact descriptions of past behaviors for many expert problem solvers. But, even for less experienced problem solvers, these same heuristics also were expected to provide useful answers to the question: “What should I do when I’m stuck?”

Unfortunately, for reasons we describe briefly in this, our first of two papers for ICME 11, the past 50 years of research have not provided validation for these latter expectations. Nonetheless, some hope remains! Most past research has leaped ahead to investigate the questions: (a) Can Polya-style heuristics be taught? (b) Do learned heuristics/strategies have positive impacts on students’ competencies? There exists almost no research that has provided useful operational definitions to answer more fundamental questions such as: (a) What does it mean to “understand” Polya-style heuristics? (b) How (and in what ways) do these understandings develop? (c) What is the nature of primitive levels of development? (d) How can development be reliably observed, documented, and measured (or assessed)? Until researchers develop useful responses to these latter two questions, it is not reasonable to expect significant progress to be made on the former two questions.

In spite of the apparent face validity of Polya’s heuristics, Begel’s (1979) comprehensive review of the research literature in mathematics education concluded that there was little evidence to support the claim that general processes that experts use to describe their past problem solving behaviors also should provide prescriptions to guide novices’ next-steps. Similarly, Silver’s (1985) assessment of the literature on problem solving concluded that, even in studies where some successful learning has been reported, transfer of learning has been unimpressive. Furthermore, successes generally occurred only when world-class teachers taught long and complex courses in which the size and complexity of the “treatments” made it unclear why performance improved. Perhaps, suggested Silver, improvements in problem solving performance simply resulted from students learning relevant mathematics concepts - rather than from learning problem solving strategies, heuristics, or problem solving processes!

Similar conclusions again were stated in the NCTM’s 1992 Handbook for Research on Mathematics Teaching and Learning (Grouws, 1992), where Schoenfeld’s (1992) chapter on problem solving concluded that attempts to teach students to use Polya-style heuristics and processes generally had not proven to be successful. However, Schoenfeld went on to suggest that one reason for this lack of success might be because many of Polya’s heuristics appear to be descriptive but not prescriptive. That is, most are really just names for large categories of
processes rather than being well defined processes in themselves. Therefore, in an attempt to go beyond “descriptive power” to achieve “prescriptive power,” Schoenfeld suggested that problem solving research and teaching should: (a) Help students develop larger numbers of more specific problem solving strategies that link more clearly to specific classes of problems, (b) Teach metacognitive strategies1 so that students learn when to use their problem solving strategies and content knowledge, and (c) Develop ways to improve students’ beliefs about the nature of mathematics, problem solving, and their own personal competencies.

Unfortunately, ten years after Schoenfeld’s proposals were made, Lester and Koehle (2003) again reviewed the literature and again concluded that research on problem solving still had little to offer to school practice. One explanation for this lack of success appeared to be because Schoenfeld’s proposal simply moved the basic shortcoming of Polya’s heuristics to a higher level. That is, regardless of whether attention focuses on Polya-style heuristics or on Schoenfeld-style metacognitive processes or beliefs, short lists of descriptive processes or rules tend to be too general to have prescriptive power. Yet, longer lists of prescriptive processes or rules tend to become so numerous that knowing when to use them becomes the heart of what it means to understand them. This shortcoming tends to be exacerbated by the fact that regardless of whether attention focuses on Polya’s heuristics or on Schoenfeld’s metacognitive processes or beliefs, virtually all such processes and rules are counterproductive in some situations. For example, even the seemingly-sensible admonition for students to carefully plan-monitor-assess their work tends to be explicitly set aside during periods of productive “brainstorming” during initial stages of solving complex problems. In fact, the defining characteristic of brainstorming is that problem solvers are supposed to rapidly generate a diverse collection of ideas – by temporarily avoiding criticism, assessment, and concerns about long-range implications. So, knowing when and why to use such techniques emerges as one of the most important parts of what it means to understand them.

In response to such conclusions about the state of problem solving research, Schoenfeld’s plenary address for the 2007 NCTM Research Pre-session proposed another embellishment of his same basic theory. The heart of his recommendation was that researchers should focus on something that we might call meta-meta-cognitive processes – or rules which are expected to operate on lower-level metacognitive processes, heuristics, strategies, knowledge, or skills. But again, just as in the case of the beliefs and metacognitive processes that Schoenfeld proposed fifteen years earlier, meta-meta-cognitive processes were described as being explicitly executable rules (e.g., cost-benefit rules which operate on lower-level rules). Consequently, it is unclear why meta-metacognitive rules should be expected to avoid the same shortcomings that were associated with past notions of heuristics, strategies, or meta-cognitive processes. That is, short lists of descriptive rules lack prescriptive power, and longer lists of prescriptive rules involve knowing when and why to use them.

Lesh and Zawojewski (2007) concluded that, when a field of research has experienced more than 50 years of failure using continuous embellishments of rule-governed conceptions of problem solving competence, perhaps the time has come to consider other options – and to re-examine foundation-level assumptions about what it means to understand mathematics concepts and problem solving processes. In particular, it is time to re-examine foundation-level assumptions about what it means to understand a small number of big ideas in elementary mathematics. One alternative is to use theoretical perspectives and accompanying research methodologies that we call models & modeling perspectives (MMP) on mathematics problem solving, learning, and teaching (Lesh & Doerr, 2003). But, before describing

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1 Metacognitive processes are processes that operate on lower-order knowledge or abilities.
relevant aspects of MMP, we briefly identify some of the major reasons why past problem solving research has produced so little success.

LIMITING FACTORS IN PROBLEM-SOLVING RESEARCH

Pendulum Swings Fuelled by High-Stakes Testing

Over the past several decades, we have seen numerous cycles of pendulum swings between a focus on problem solving and a focus on “basic skills” in school curricula. These approximately 10-year cycles, especially prevalent in the USA but also evident in other nations, appear to have brought few knowledge gains with respect to problem solving development from one cycle to the next (Lesh & Zawojewski, 2007). Over the past decade or so, many nations have experienced strong moves back towards curricula materials that have emphasized basic skills. These moves have been fuelled by high-stakes national and international mathematics testing, such as PISA (Programme for International Student Assessment: http://www.pisa.oecd.org/) and TIMMS (Third International Mathematics and Science Study: http://timss.bc.edu/timss2003i/intl_reports.html).

These test results have led many nations to question the substance of their school mathematics curricula. Indeed, the strong desire to lead the world in student achievement has led several nations to mimic curricula programs from those nations that score highly on the tests, without well-formulated plans for meeting the specific needs of their student and teacher populations (Sriraman & Adrian, 2008). This teaching-for-the test has led to a “New Push for the Basics” as reported in the New York Times, November 14, 2006. Unfortunately, these new basics are not the basics needed for future success in the world beyond school, as we indicate later. With this emphasis on basic skills, at the expense of genuine real-world problem solving, the number of articles on research in problem solving has declined. What is needed is research that explores students’ concept and skill development as it occurs through problem solving.

Limited Research on Concept Development and Problem Solving

As we discuss in our second paper, relationships are unclear between concept development and the development of problem solving competencies (Lester & Charles, 2003; Schoen & Charles, 2003). One shortcoming of past problem solving research is that it has not been clear how concept development is expected to interact with the development of relevant problem solving heuristics, beliefs, dispositions, or processes. In fact, in many curriculum standards documents (e.g., NCTM, 2000, 2008 http://standards.nctm.org/document/chapter3/prob.htm), problem solving tends to be listed as the name of a chapter-like topic similar to algebra, geometry, or calculus. In other words, the implicit assumption is conveyed that problem solving ability is expected to increase by: (a) first, mastering relevant concepts, (b) second, mastering relevant problem solving heuristics, strategies, beliefs, dispositions, or processes, and (c) third, learning to put these concepts and processes together to solve problems. Consequently, when such assumptions are coupled with the flawed belief that students must first learn concepts and processes as abstractions before they can put them together and use them in “real-life” problem solving situations, problem solving tends to end up never getting taught at all in many classrooms. So, one of the most critical challenges for future problem solving research is to clarify the nature of relationships that should exist between concept development and the development of problem solving competencies.

Limited Knowledge of Students’ Problem Solving Beyond the Classroom
As we have highlighted, problem solving is a complex endeavor involving, among others, mathematical content, strategies, thinking and reasoning processes, dispositions, beliefs, emotions, and contextual factors. Future studies of problem solving need to embrace the complexity of problem solving as it occurs in school and beyond, as we discuss later. However, to date, most research on problem solving has not really addressed students' problem solving capabilities beyond the classroom—we need to know why students have difficulties in applying the mathematical concepts and abilities (that they presumably have learned in school) outside of school—or in other classes such as those in the sciences. To assist us here we need more interdisciplinary problem solving experiences that mirror problem solving beyond the classroom (English, in press). For example, experiences that draw upon the broad field of engineering provide powerful links between the classroom and the real world, enabling students to apply their mathematics and science learning to the solution of authentic problems (Kuehner & Mauch, 2006).

Changing Nature of the Types of Problem Solving and Mathematical Thinking needed beyond School

Today, experts outside of schools consistently emphasize that new technologies for communication, collaboration, and conceptualization have led to significant changes in the kinds of mathematical thinking that are needed beyond school—and to significant changes in the kinds of problem solving situations in which some form of mathematical thinking is needed. For example, in just a few decades, the application of mathematical modeling to real-world problems has escalated. Traffic jams are modeled and used in traffic reports; the placement of cell-phone towers is based on mathematical models involving 3-D topography of the earth; and the development of internet search engines is based on different mathematical models designed to find new and more efficient ways to conduct searches. Unfortunately, the types of problems students meet in the classroom are often far removed from reality—we need to redress this state of affairs as we consider fresh perspectives on problem solving in the curriculum.

Research on problem solving beyond school also suggests that, although professionals in mathematics-related fields draw upon their school learning, they do so in a flexible and creative manner, unlike the way in which they experienced mathematics in their school days (Gainsburg, 2006; Hall, 1999; Hamilton, 2007; Noss, Hoyles, & Pozzi, 2002; Zawojewski & McCarthy, 2007). Furthermore, problem solvers beyond the classroom often are not isolated individuals but instead are teams of diverse specialists (Hutchins, 1995a, 1995b; Sawyer, 2007). These specialists often offload important aspects of their thinking using powerful technology-based tools which make some functions easier (such as information storage, retrieval, representation, or transformation) but which make others far more complex and difficult (such as interpretation and communication). So, relevant knowledge and abilities tend to be distributed across a variety of tools, and across individuals within groups. Critical abilities often are those associated with the mathematics of description, explanation, and communication at least as much as the mathematics of computation and deduction (Lesh, Middleton, Caylor & Gupta, 2008), and progress tends to resemble the evolution of a community or interacting organisms – rather than movement along a path (Lesh & Yoon, 2004). Unfortunately, research on mathematical problem solving has not kept pace with the rapid changes in the mathematics and problem solving needed beyond school.

Lack of Accumulation of Problem solving Research

As we also discuss in our second paper, there has been a lack of accumulation of problem solving research. Failed or flawed concepts or conjectures have continued to be recycled or embellished – with no significant changes being made in the underlying theoretical
perspectives. Mathematics education researchers have generally avoided tasks that involve developing critical tools for their own use. Unlike their counterparts in more mature sciences (physics, chemistry, biology), where some of the most significant kinds of research often involve the development of tools to reliably observe, document, or measure the most important constructs, mathematics educators have developed very few tools for observing, documenting, or measuring most of the understandings and abilities that are believed to contribute to problem solving expertise. We return to this concern later in this paper.

Furthermore, partly because operational definitions and tools have not been developed for most constructs that have been considered important in problem solving development, there is a tendency to repeatedly elaborate on or recycle apparently failed or flawed concepts. For example, the use of Polya-style heuristics, problem solving strategies, and various metacognitive and meta-metacognitive processes (Schoenfeld, 2007) is an example of continuous embellishment of a theory that focuses on explicitly learned rules. In our second paper, we extend our discussion on theory development and explore alternative research methodologies for advancing the field.

ADVANCING THE FIELD OF PROBLEM SOLVING RESEARCH AND CURRICULUM DEVELOPMENT

Although we have highlighted some of the issues that have plagued problem solving research, there are emerging signs that the situation is starting to improve. We believe the pendulum of change is beginning to swing back towards problem solving on an international level, providing impetus for new perspectives on the nature of problem solving and its role in school mathematics (Lester & Kehle, 2003). For example, a number of Asian countries have recognized the importance of a prosperous knowledge economy and have been moving their curricular focus toward mathematical problem solving, critical thinking, creativity and innovation, and technological advances (e.g., Maclean, 2001; Tan, 2002). In refocusing our attention on problem solving and how it might become an integral component of the curriculum rather than a separate, often neglected, topic we explore the following issues:

- What is the nature of problem solving in various arenas of today’s world?
- What future-oriented perspectives are needed on the teaching and learning of problem solving including a focus on mathematical content development through problem solving?
- How can studies of problem solving expertise contribute to theory development that might guide the design of worthwhile learning experiences?
- Why is a models and modeling perspective a powerful alternative to existing views on problem solving?

The Nature of Problem Solving in Today’s World

Concerns have been expressed by numerous researchers and employer groups that schools are not giving adequate attention to the understandings and abilities that are needed for success beyond school. For example, potential employees most in demand in mathematics/science related fields are those that can (a) interpret and work effectively with complex systems, (b) function efficiently and communicate meaningfully within diverse teams of specialists, (c) plan, monitor, and assess progress within complex, multi-stage projects, and (d) adapt quickly to continually developing technologies (Lesh, in press). Research indicates that such employees draw effectively on interdisciplinary knowledge in solving problems and communicating their findings. Furthermore, although they draw upon their school learning, these employees do so in a flexible and creative manner, often creating or reconstituting
mathematical knowledge to suit the problem situation, unlike the way in which they experienced mathematics in their school days (Gainsburg, 2006; Hamilton, 2007; Lesh, in press; Zawojewski & McCarthy, 2007). In fact, these employees might not even recognize the relationship between the mathematics they learned in school and the mathematics they apply in solving the problems of their daily work activities.

Identifying and understanding the differences between school mathematics and the workplace is critical in formulating a new perspective on problem solving. As we address later, one of the notable findings of studies of problem solving beyond the classroom is the need to master mathematical modeling. Many new fields, such as nanotechnology, need employees who can construct basic yet powerful constructs and conceptual systems to solve the increasingly complex problems that confront them. Being able to adapt previously constructed mathematical models to solve emerging problems is a critical component here.

Future-Oriented Perspectives on the Teaching and Learning of Problem Solving

We have argued that future-oriented perspectives on problem solving should transcend current school curricula and national standards and should draw upon a wider range of research across disciplines (English, 2008; Lesh, in press). Most research on problem solving has commenced with the assumption that the researchers already possess clear and accurate understandings about what it means to "understand" problem solving. This is not necessarily the case, as we have indicated (e.g., retrospective descriptions of observed problem solving do not necessarily provide useful forward-looking prescriptions for what problem solvers should do as "next steps" during problem solving sessions).

A critical component of any agenda to advance the teaching and learning of problem solving is the clarification of the relationships and connections between the development of mathematical content understanding and the development of problem solving abilities, as we have emphasized earlier in this paper. If we can clarify these relationships we can inform curriculum development and instruction on ways in which we can use problem solving as a powerful means to develop substantive mathematical concepts. In so doing, we can provide some alternatives to the existing approaches to teaching problem solving. These existing approaches include instruction that assumes the required concepts and procedures must be taught first and then practiced through solving routine "story" problems that normally do not engage students in genuine problem solving (primarily a content-driven perspective). Another existing approach, which we have highlighted earlier, is to present students with a repertoire of problem solving heuristics/strategies such as "draw a diagram," “guess and check,” “make a table” etc. and provide a range of non-routine problems to which these strategies can be applied (primarily a problem solving focus). Unfortunately, both these approaches treat problem solving as independent of, or at least of secondary importance to, the concepts and contexts in question.

A powerful alternative to these approaches is one that treats problem solving as integral to the development of an understanding of any given mathematical concept or process. This perspective (Lesh & Zawojewski, 2007) also reflects the recognition that the problem solving of novices and experts differs in ways that go beyond their observed behaviors, that is, what they actually do in solving a problem. Novices and experts see (interpret and re-interpret) problem situations differently—experts focus on the underlying structural features of a problem situation so for them, problem solving involves an interplay between problem structure (content) and problem solving processes. We continue this discussion in the next section.
Studies of Problem solving Expertise and their Contributions to Theory Development

The seminal work of Krutetskii (1976) has shown how gifted mathematics students have a repertoire of ideas, strategies, and representations that seem to be organized into a highly sophisticated network of knowledge, equipping them with powerful ways to approach problem solving situations. As noted above, experts readily perceive the underlying structures of problem situations, project ahead to remove unnecessary steps in the solution process, and are able to generalize broadly. When we explore expert problem solving beyond the classroom, we see other factors that play a key role. For example, the knowledge of experts in workplace environments that require heavy use of mathematics tends to be more organized around the mathematics of the situation than around general problem solving strategies or traditional mathematical topics (e.g., Gainsburg, 2006; Hall, 1999).

Although such studies have provided rich insights into how experts perform in given problem solving situations, they do “not guarantee that one is studying the experts at what actually makes them experts” (Lester & Kehle, 2003, p. 504). In other words, how do experts become experts? We need new studies on the nature and development of expertise—how expertise evolves within episodes of problem solving and over many experiences. Presumably, students’ understandings of problem solving are not so different from their understandings of other aspects they are to learn in mathematics. For example, students’ understandings of problem solving heuristics probably develop. And, development should be able to be traced. So, we need more studies about problem solving that are similar to the studies that mathematics educators have conducted about the development of concepts and abilities in topic areas such as early number concepts, rational number concepts, early algebra concepts, and so on.

Rather than just describe the behavior we observe as experts solve problems, we need to know how they interpret the problem situations, how they mathematize them, how they quantify them, how they operate on quantities, and so on (Lesh & Zawojewski, 2007). Furthermore, we need to look beyond the assumption that experts initially learn content, then acquire problem solving strategies, and then learn ways to apply the mathematics and strategies they have developed. As Zawojewski and Lesh (2003) and others have argued, the development of problem solving expertise appears as a synergistic, holistic development of varying degrees of mathematical content, problem solving heuristics/strategies, higher-order thinking, and affect—all of which are situated in particular contexts.

Theory Development: A Models and Modeling Perspective (MMP) on the Development of Problem Solving in and beyond the Classroom

Before we explore theory development, we need to offer a more appropriate definition of problem solving, one that does not separate problem solving from concept development as it occurs in real-world situations beyond the classroom. We adopt here the definition of Lesh and Zawojewski (2007):

A task, or goal-directed activity, becomes a problem (or problematic) when the “problem solver” (which may be a collaborating group of specialists) needs to develop a more productive way of thinking about the given situation (p. 782).

Thinking in a productive way requires the problem solver to interpret a situation mathematically, which usually involves progression through iterative cycles of describing, testing, and revising mathematical interpretations as well as identifying, integrating, modifying, or refining sets of mathematical concepts drawn from various sources (Lesh & English, 2005; Lesh & Zawojewski, 2007). These processes are the rudiments of mathematical modeling. Seeing problem solving from a models and modeling perspective
(MMP) contrasts with the traditional definition of problem solving as searching for a way to progress from the “givens” to the “goals.” Rather, from a models and modeling perspective, problem solving involves iterative cycles of understanding the givens and the goals of a problem. In the remainder of this section we highlight some of the key features of problem solving from a models and modeling perspective.

1. When the solution to a problem involves the development of a model (or artifact or conceptual tool), and when the underlying conceptual systems are expressed in forms that can be examined and assessed by students themselves, solutions tend to involve sequences of iterative express-test-revise cycles similar to the kind that are involved in the first-, second-, and nth-drafts in the development of other kinds of symbolic or graphic descriptions of situations. Furthermore, if the underlying conceptual system is one that Piaget-inspired researchers have investigated, then the modeling cycles that problem solvers go through during a single 60-90 minute problem solving session are often strikingly similar to the stages of development that Piagetians have documented over time periods of several years. Consequently, we have sometimes referred to such sessions as local conceptual development sessions (Lesh & Harel, 2003) – because students’ thinking often evolves through several stages similar to those recognized by the Piagetians during a single 60-90 minute episode.

2. When significant conceptual adaptations occur within a single problem solving session, researchers are able to go beyond observing sequential states of knowledge to also directly observing processes that lead from one state to another. And, such observations have made it clear that it is seldom appropriate to think of solution processes as activities in which students connect previously-mastered-but-disconnected concepts and processes. Nor do solutions involve movement along a path which is formed by linking together concepts, processes, facts, and skills. Instead, problem solvers’ early interpretations tend to involve a collection of partly-overlapping-yet-undifferentiated partial interpretations of different aspects of the situations of conceptual systems. So, regardless of whether the problem solver is an individual or a group, model development tends to involve gradually sorting out, clarifying, revising, refining, and integrating conceptual systems that are at intermediate stages of development.

3. When solutions to problems involve the development of mathematically significant artifacts or tools, when the underlying design is an important part of the product that is designed to help solve the problem, and when the product needs to be powerful (for the specific situation in which it was first created), sharable (with other people), and reusable (in situations different to the one in which it was first created), the knowledge and abilities that are embodied in these products tend to be generalizable and transferable.

4. Heuristics that are intended to help problem solvers make productive adaptations to existing ways of thinking tend to be significantly different from heuristics that are intended to help problem solvers figure out what to do when they are stuck (with no apparent concepts available). Their functions tend to have far less to do with helping students know what to do next, and have far more to do with helping them interpret the situation (including alternative ways of thinking about givens, goals, personal competencies, and “where they are” in solution processes). Furthermore, such heuristics often function tacitly rather than as explicitly executed rules. So, learning them is similar to situations in which athletes or performing artists analyze videotapes of their own performances (or those of others). It is useful to develop languages (interpretation systems and conventions for making interpretations) for describing these past performances. But, such languages usually are not

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2 Here, we use the term “embodiment” in the way used by Zoltan Dienes who introduced the notion of concrete embodiments of mathematical concepts (Dienes, 1960).
intended to give rise to prescriptive rules about what to do at specific points in future performances. *Instead, the language and imagery tends to be aimed mainly at helping students make sense of things during future performances. In other words, they are aimed mainly at the development of more powerful models.*

5. When problem solving involves model development, *heuristics and metacognitive processes tend to evolve in ways that are quite similar to the dimensions of development that apply to other types of concepts or abilities that mathematics educators have studied.* For example, Vygotsky’s (1978) concept of internalizing external functions often results in early understandings of heuristics that are distinctly social in character. So, instead of “looking at a similar problem” students may find it more useful to think of themselves as “looking at the same problem from another point of view” (and to be aware of the fact that one’s current point of view is not the only possible point of view).

6. In fields like engineering it is considered to be “common knowledge” that realistic solutions to realistically complex problem solving situations nearly always need to integrate concepts and procedures drawn from a variety of textbook topic areas or theories. Likewise, when students develop realistically useful models (or other conceptual tools) for making sense of realistically complex “real-life” situations, *they often need to integrate ideas and procedures drawn from more than a single textbook topic area* (measurement, geometry, probability, statistics, and algebra). One reason for this is because useful solutions often involve *trade-offs* involving conflicting goals associated with multiple agents. These goals may involve low costs but high quality, or low risk but high gain, or rapid but thorough development. The models that are produced are “chunks of knowledge” that represent inherently connected ideas that need to be unpacked in follow-up teaching and learning activities. Even after connected ideas are unpacked, *students’ knowledge often continues to be organized around experience as much as it is organized around abstractions.*

7. When problem solvers describe or design things mathematically, they tend to do more than simply engage logical-mathematical systems; they also engage feelings, values, beliefs, and a variety of problem solving processes, facts, and skills. So, *the development of processes, skills, attitudes, beliefs, is part of the development of specific models.* Skills, attitudes, and beliefs are not developed separately in the abstract before they are connected to concepts or conceptual systems; skills, attitudes, and beliefs are engaged and developed when the relevant models are engaged. Thus, skills, attitudes, and beliefs are integral parts of relevant models.

**CONCLUDING POINTS**

We have argued in this paper that research on mathematical problem solving has stagnated for much of the 1990s and early part of this century. Furthermore, the research that has been conducted does not seem to have accumulated into a substantive, future-oriented body of knowledge on how we can effectively promote problem solving within and beyond the classroom. This lack of progress is mainly due to the many years of repeated elaborations of rule-governed conceptions of problem solving competence.

The time has come to consider other options for advancing problem solving research and curriculum development—we have highlighted the need to re-examine foundation-level assumptions about what it means to understand mathematics concepts and problem solving processes. One powerful alternative we have advanced is to utilize the theoretical perspectives and accompanying research methodologies of a *models & modeling perspective* (MMP) on mathematics problem solving, learning, and teaching. Our second paper elaborates further on this perspective and on the associated research methodologies.
Adopting an MMP means researchers who study students’ models and modeling developments naturally utilize integrated approaches to exploring the co-development of mathematical concepts, problem solving processes, metacognitive functions, dispositions, beliefs, and emotions. These researchers also view problem solving processes developmentally, in a similar way they would in studying the development of mathematical concepts in topic areas such as early number, geometry, and algebra. In addition, the problems used are simulations of appealing, authentic problem solving situations (e.g., selecting sporting teams for the Olympic Games) and engage students in mathematical thinking that involves creating and interpreting situations (describing, explaining, communication) at least as much as it involves computing, executing procedures, and reasoning deductively.

REFERENCES


TEACHING MATHEMATICS IN THE CLASSROOM THROUGH PROBLEM SOLVING

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ABSTRACT

We present an approach to teaching-learning-evaluating mathematics through problem solving. The historical context is briefly described leading up to the current guidelines of the National Council of Teachers of Mathematics (USA), on which we base a characterization of teaching through problem solving. Considered a teaching method, its foundations and general guidelines for implementation in the classroom are presented. Mathematics teaching-learning-evaluation through problem solving has been used and studied systematically, at all educational levels and in teacher education activities, by the Problem Solving Work and Study Group (Grupo de Trabalho e Estudos em Resolução de Problemas – GTERP), based at UNESP, Rio Claro, São Paulo, Brazil. The research developed by the group follows essentially qualitative approaches, with the main objective of reflecting on and analyzing the possibilities this method offers for increasing learning and improving teaching processes, as well as promoting improvement of the practices of mathematics teachers. Our studies show that students’ construction of knowledge related to mathematical concepts and contents is more meaningful and effective, and when applied in teacher education activities, it favors significant improvements in their teaching practice.

INTRODUCTION

Mathematical problems have occupied a central position in school mathematics curricula since Antiquity. Records of mathematical problems can be found in ancient Egyptian, Chinese, and Greek history, and problems are still found in mathematics textbooks from the 19th and 20th Centuries, and up to the current day. According to Stanic and Kilpatrick (1989), the main point to consider in the examples presented in these books is they reflect a very limited view of learning and problem solving. The role of problem solving in the school mathematics curriculum is the result of conflicting forces linked to old and enduring ideas regarding the benefits of studying mathematics and a variety of events that occurred at the beginning of the 20th Century.
Problem solving as a classroom teaching method, here denoted Mathematics Teaching-Learning-Evaluation through Problem Solving, is a fairly new concept in mathematics education, despite the long history of problem solving in school mathematics. Consequently, the method has not been the object of very much research.

**Problem Solving and Research in the 20th Century**

The definition presented by Leder (1998) for “educational research” outlines the various aspects that it can address, including: the purposes of education, teaching and learning processes, professional development, organizational resources, policies, and strategies. The volume of studies on library shelves indicates that it has become an immense enterprise, and that the search for new knowledge not only continues, but has been widely documented. In recent years, the volume, scope, and diversity of educational research, in general, and research in mathematics education, in particular, have grown substantially.

Having begun as a field of systematic studies with the work of Felix Klein at the beginning of the 20th Century, mathematics education had grown into a vast and intricate endeavor by the end of the century. Felix Klein was one of the most important mathematicians of the late 19th Century, and one of the last, together with Gauss, Riemann, and Poincaré, to break the barrier of specialization and provide the fundamental elements that gave impetus to the mathematics of the 19th and early 20th Century. It was then that he wrote his book *Elementary Mathematics from an Advanced Standpoint.*

Klein stated, in his autobiography written in 1923, that the totality of all knowledge and the ideal of a complete education should not be neglected because of specialized studies, and that universities should be concerned with preparatory teaching in the schools. He emphasized teacher education, in particular. He was a brilliant mathematician who was sincerely and seriously concerned about issues related to teaching.

At that time, mathematics teaching was characterized by work based on repetition, where the memorization of facts was considered important. Years later, a different orientation began to emphasize that students should learn with comprehension, and should understand what they were doing.
It was then that talk began of solving problems. George Polya (1944) emerged as a reference, emphasizing the importance of discovery and of encouraging students to think by means of problem solving. In his book *How to Solve It*, he states “A great discovery solves a great problem, but there is always a bit of discovery in the solution of any problem”. In 1949, he wrote that solving problems is the specific realization of intelligence, and that if education does not contribute to the development of intelligence, it is obviously incomplete. In 1948, the work developed by Herbert F. Spitzer in basic arithmetic, in the U.S.A., was based on learning with comprehension, always using problems-situations; and in 1964, in Brazil, the teacher Luis Alberto S. Brasil defended teaching mathematics using problems that generated new concepts and contents.

In the 1960s and 1970s, mathematics teaching in Brazil and in other countries of the world was influenced by a reform movement known as Modern Mathematics. This reform dominated the scene, and like the others before it, failed to include the participation of classroom teachers. It presented a mathematics based on structures of logic, algebra, topology, and order, and emphasized set theory. It highlighted many properties, reflected excessive concern with mathematical abstractions, and used a universal, precise, and concise language. However, it emphasized the teaching of symbols and complex terminology, which compromised learning. In this reform, teaching was approached with excessive formalization, distancing itself from practical issues.

According to Onuchic & Allevato (2005), these reforms were not as successful as expected. The questions continued: Are these reforms aimed at preparing a citizen who is useful to the society in which he/she lives? Do they seek to teach mathematics in a way that prepares students for a world of work that demands mathematical knowledge? In addition to this, particularly in the 1970s, there was a growing concern with a mathematics curriculum that was initially aimed at increasing test scores, also known as computational ability tests.

Concomitantly, at the beginning of the 1970s, systematic investigation of problem solving and its implications for curricula was initiated. Thus, the importance attributed to problem solving is relatively recent, and only in this decade did mathematics educators come to accept the idea that the development of problem-solving abilities deserved more attention. At the end of the 1970s, problem solving emerged, gaining greater acceptance around the world. In
1976, at the 3rd International Congress on Mathematical Education, in Karlsruhe, Germany, problem solving was one of the themes addressed.

Discussions in the field of mathematics education in Brazil and around the world demonstrated the need to adapt school work to new trends that could lead to improved ways of teaching and learning mathematics.

In the U.S.A., the National Council of Teachers of Mathematics (NCTM) responded to this concern with a series of recommendations for the progress of school mathematics, published in 1980 in a document entitled An Agenda for Action (NCTM, 1980). All interested groups and individuals were called to collaborate in the work to seek, through a massive cooperative effort, a better mathematical education for all. The first recommendation in the document was that “problem solving be the focus of school mathematics in the 1980s.”

During the 1980s, many resources were developed to facilitate work with problem solving in the classroom, such as collections of problems, lists of strategies, suggestions for activities, and guidelines to evaluate student work involving problem solving. Much of this material contributed to helping teachers make problem solving the central point of their work.

Nevertheless, possibly due to differences in conceptions regarding the significance of problem solving becoming “the focus of school mathematics”, the work during that decade failed to achieve a good level of progress (ONUCHIC, 1999). Schroeder & Lester (1989) presented three different approaches to problem solving that help us reflect on these differences: theorizing about problem solving; teaching how to solve problems; and teaching mathematics through problem solving. Teachers who teach about problem solving seek to emphasize Polya’s model, or a derivation of it. When teaching how to solve problems, the teacher concentrates on the manner in which mathematics is taught, and what of this can be applied in the resolution of routine and non-routine problems. In this view, the essential purpose for learning mathematics was to be able to use it. As the 1980s ended with all these recommendations for action, researchers began to question the teaching and the effect of strategies and models, and to discuss the didactic-pedagogical perspectives of problem solving.

Problem solving, as a teaching method, became a focus of research and studies in the 1990s. This new view of mathematics teaching-learning was based especially on studies developed by the NCTM that culminated in the publication of the Standards 2000, officially Principles and Standards for School Mathematics.(NCTM, 2000). Problem solving was
emphasized as one of the standards for the process of teaching mathematics, and teaching through problem solving was strongly recommended. (ONUCHIC; ALLEVATO, 2005).

Drawing on the ideas of the NCTM Standards, the PCN (National Curriculum Parameters) were created in Brazil (BRASIL; 1997, 1998, 1999), which pointed to the development of the capacity to solve problems, explore them, generalize from them, and even propose new problems based on them, as one of the purposes of mathematics teaching. They indicated problem solving as the point of departure for mathematics activities, and discussed ways to do mathematics in the classroom.

Today, at the beginning of the 21st Century, some of the greatest challenges faced by mathematics educators in past decades have persisted, changed, or proliferated, as teaching and society have grown more complex. In “Unfinished Business: Challenges for Mathematics Educators in the Next Decades”, Kilpatrick & Silver (2000) outline what they believe to be the main challenges: guarantee mathematics for all; promote students’ understanding; maintain balance in the curriculum; use evaluation as an opportunity for learning; and develop professional practice.

Cai (2003) emphasizes, however, that although little is known regarding how students attribute meaning and learn mathematics through problem solving, many ideas associated with this approach – the change in the teacher’s role, the selection and elaboration of problems, collaborative learning, among others – have been researched intensively, offering answers to various frequently-asked questions regarding this way of teaching.

**MATHEMATICS TEACHING-LEARNING-EVALUATION THROUGH PROBLEM SOLVING**

The compound word “teaching-learning-evaluation” was chosen intentionally to express the idea that teaching and learning should take place simultaneously during the construction of knowledge, with the teacher as guide and the students as co-builders of this knowledge. In addition, this methodology integrates a more updated conception regarding evaluation. It is constructed during the problem solving, integrated with the teaching to follow students’ growth, increasing learning and reorienting practice in the classroom when necessary.

According to Van de Walle (2001), truly effective mathematics teachers should involve four basic components in their work: (1) an appreciation of the discipline of mathematics itself,
meaning “doing mathematics”; (2) an understanding of how students learn and construct ideas; (3) the ability to plan and select tasks so that students learn mathematics in a problem solving environment; (4) the ability to integrate evaluation with the process to increase learning and improve teaching from day to day.

Teaching-Learning-Evaluation of Mathematics through Problem Solving differs from approaches that privilege rules regarding “how to”. It “reflects a tendency to react to past characteristics, like a set of facts, mastering algorithmic procedures, or the acquisition of knowledge through routine or mental exercise”. (ONUCHIC, 1999, p.203).

It corresponds to work in which a problem is the point of departure for learning, and the construction of knowledge occurs in the process of solving it. Teacher and students develop the work together, and learning takes place collaboratively in the classroom (ALLEVATO, ONUCHIC, 2007; ONUCHIC; ALLEVATO, 2005). The methodology is similar to the Japanese approach to teaching mathematics through problem solving. In Problem Solving as a Vehicle for Teaching Mathematics: a Japanese Perspective, Yoshinori Shimizu (2003) writes that “Japanese teachers in elementary schools often organize an entire mathematics lesson around multiple solutions to a single problem in a whole-class instructional mode. This organization is particularly useful when introducing a new concept or a new procedure during the initial phase of a teaching unit.”(p.206).

For Van de Walle (2001), a problem is any task or activity for which students have no prescribed or memorized methods or rules, and no perception that a specific method for arriving at the correct solution exists. Adding a subjective character to this question, in the context of the methodology presented here, we consider that problem refers to “everything that we do not know how to do, but are interested in doing”.

There are no rigidly defined ways to put this methodology into practice (SHIMIZU, 2003; KRULIK; RUDNICK, 2005; ONUCHIC; ALLEVATO, 2005; VAN DE WALLE; LOVIN, 2006). One proposal is to organize activities according to the following stages:

1) Form groups and hand out the activity. The teacher presents the problem to the students, who, divided into small groups, read and try to interpret and understand the problem. It should be emphasized that the mathematical content necessary, or most appropriate, to solve the problem has not yet been presented in class. The problem proposed to the students, which we call
the generative problem, is what will lead to the content that the teacher plans to construct in that lesson.

2) Observe and encourage. The teacher no longer has the role of transmitter of knowledge. While students attempt to solve the problem, the teacher observes, analyzes students’ behavior, and stimulates collaborative work. The teacher mediates in the sense of guiding students to think, giving them time to think, and encouraging the exchange of ideas among students.

3) Help with secondary problems. The teacher encourages students to use their previous knowledge, or techniques that they already know, to solve the problem, and stimulates them to choose different methods based on the resources they have available. Nevertheless, it is necessary to assist students with their difficulties, intervening, questioning, and following their explorations, and helping them to solve secondary problems when necessary. These refer to doubts presented by the students in the context of the vocabulary present in the statement of problem; in the context of reading and interpretation; as well as those that might arise during the problem solving, e.g. notation, the passage from vernacular to mathematical language, related concepts, and operational techniques, to enable the continuation of the work.

4) Record solutions on the blackboard. Representatives of the groups are invited to record solutions on the blackboard. Correct as well as incorrect solutions, as well as those done for different processes, should be presented for all the students to analyze and discuss.

5) Plenary session. The teacher invites all students to discuss solutions with their colleagues, to defend their points of view and clarify doubts. The teacher acts as a guide and mediator in the discussions, encouraging the active and effective participation of all students, as this is the richest moment for learning.

6) Seek consensus. After addressing doubts and analyzing resolutions and solutions obtained for the problem, the teacher attempts to arrive at a consensus with the whole class regarding the correct result.

7) Formalize the content. At this moment, called “formalization”, the teacher makes a formal presentation of the new concepts and contents constructed, highlighting the different operative techniques and properties appropriate for the subject.

It should be reiterated that, in this methodology, the problem is proposed to the students before the mathematical contents necessary or most appropriate for solving it (planned by the
teacher according to the program for that grade level) have been formally presented. Thus, the teaching-learning of a mathematical topic begins with a problem that expresses key aspects of this topic, and mathematical techniques should be developed in the search for reasonable answers to the problem given.

For Van de Walle (2001), problem solving should be seen as a main teaching strategy, and he points to the importance of beginning the work from the point where students are, contrary to other ways of teaching that begin from the point where the teachers are, ignoring what the students bring with them to the classroom. He goes on to state that teaching with problems has great value, and that despite the difficulties, there are good reasons to engage in the effort.

Without a doubt, teaching mathematics through problem solving is an approach consistent with the recommendations of the NCTM (NCTM, 2000) and the Brazilian National Curriculum Parameters (BRASIL, 1997, 1998, 1999), as the mathematical concepts and abilities are learned in the context of problem solving. The development of thinking processes at the university level will be promoted through these experiences, and the work of teaching mathematics will take place in an environment of investigation guided by problem solving.

In agreement with Krulik & Rudnick (2005), and always with the objective of carrying out ongoing evaluation, new problems related to the generative problem are proposed to the students following the stage of formalization, with the aim of analyzing whether or not the essential elements of the mathematical contents introduced in that lesson were understood. In our view, the understanding of mathematics by students involves the idea that understanding is essential relating. It should be emphasized that indications that a student understands, misunderstands, or fails to understand specific mathematical ideas emerge often as he/she solves a problem.

Rather than being the focus of mathematics teaching, when considered as a teaching method, problem solving makes understanding its central focus and objective. In this way, the emphasis attributed to problem solving is not removed, but its role in the curriculum is broadened. It moves beyond a limited activity to engage students in the application of knowledge, following the acquisition of certain concepts and techniques, to be a means of acquiring new knowledge as well as a process in which the students can apply previously constructed knowledge (ONUCHIC, 1999).
Mathematics has always played an important role in society. This role is more significant today, and may become even more important in the future. People do not always think mathematically, nor are they aware that, if they were to do so, they might make better decisions. This lack of awareness may be a failure of the mathematics taught as well as the way it is taught. Often the teaching of mathematics produces students with over-simplified conceptions and strategies that are excessively mechanical to resolve problems. For Hiebert & Behr (1989), rather than considering knowledge as a package that is ready and finished, teaching should encourage students to construct their own knowledge.

**Research and Scientific Production Related to Problem Solving at UNESP**

Romberg (1998) considers the objective of mathematics education to be the production of new knowledge about the teaching and learning of mathematics, and that since students learn most of “their mathematics” in school, research should identify the main classroom components that promote mathematical understanding, and point out the organizational characteristics that impede or contribute to the good functioning of those classrooms.

In this paper, we intend to highlight Mathematics Teaching-Learning-Evaluation through Problem Solving, a teaching method which has been used and studied systematically, at all educational levels and in teacher education activities, by the Problem Solving Work and Study Group (*Grupo de Trabalho e Estudos em Resolução de Problemas* – GTERP), based at UNESP, Rio Claro, São Paulo, Brazil. The group has been the generative nucleus of investigations, scientific production, and continuing education in problem solving.

The research conducted composes part of a larger, broader project whose main objective is to reflect on and analyze the possibilities offered by Mathematics Teaching-Learning-Evaluation through Problem Solving for increasing learning and improving teaching processes, as well as promoting improved practices among teachers in the mathematics classroom.

These studies, using a qualitative methodology, consist essentially of interventions in the realm of participant research and action research. Problem solving activities are elaborated by the researchers and applied in the classroom. Thus, a large part of the theses, dissertations, and other work produced by the group narrate and analyze situations involving pedagogical interventions realized by its members in the classroom or in the sphere of teacher education. The scientific
production encompasses a wide variety of mathematical contents at all educational levels – elementary, high school, and higher learning.

This set of research constitutes a wide spectrum of research possibilities in mathematics education. A description of the theses and dissertations, as well as the work carried out by the group up until 2005, can be found in Onuchic (1999) and Onuchic & Allevato (2005).

CONCLUSIONS

Considering mathematics educators are people who are professionally concerned about mathematics teaching and learning at any educational level, we can testify to their dedication and relevant production by the volume and quantity of studies in mathematics education carried out in the 20th Century. Students are currently the beneficiaries of the large variety of instructional materials created. Certainly if we compare teachers at the beginning of the century with the teachers of today, we can say that they are much better prepared pedagogically and mathematically. The majority of school mathematics curricula are richer than those at the beginning of the century. In spite of all this, the same complaints are heard today: that students do not like and do not learn mathematics well enough; that teachers do not know mathematics and do not know how to teach it; that school curricula are superficial, repetitive, and fragmented… All these complaints, and data obtained from other sources (research, evaluations, etc), suggest that students leave school poorly prepared, not knowing how to make use of the mathematics they worked with throughout their many years of schooling. As we already said, people are often unable to make decisions in life. These people do not always think mathematically, nor are they aware that, if they did, they would make better decisions.

The teaching method presented here constitutes a way of working in the classroom using generative problems as a point of departure. Using Mathematics Teaching-Learning-Evaluation through Problem Solving, students’ construction of knowledge related to mathematical concepts and contents occurs more meaningfully and effectively. Experiences in research with students and teacher education activities, in which this approach was used, have favored significant advances in the understanding of mathematical concepts and contents and improvements in teaching practice.
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THE METHOD OF PROBLEM SOLVING BASED ON THE JAPANESE AND POLYA’S MODELS.

A CLASSROOM EXPERIENCE IN CHILEAN SCHOOLS. ¹

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".. Only major discoveries allow us to solve major problems .. There is, a little bit of discovery in the solution of any problem, and if any problem is solved and gets to excite our curiosity, this kind of experience, at a certain age, may determine the pleasure of intellectual work and leave, both in spirit and in the character, a trace that will last for a lifetime .. "Polya (1945).

ABSTRACT

This paper is framed within the Project of Improvement of the Teaching of Mathematics in Chile. According to the agreement with the Japanese International Cooperation Agency (JICA), and within the program of International Cooperation between the Chilean and Japanese Governments. A research applied on students of Primary Education at the County of Pelarco is presented due to the pretty low results in the national and international evaluations. Students are exposed to problem solving. In order to develop this project, a unit using the Japanese Lesson Planning, Polya Model was designed to be implemented inside the classroom. The methodology was of the quantitatively and qualitatively type, through an interpretative analysis of the content. Categories were established according to the Polya 4 stages categorization model, thus allowing to analyze the outcomes of the students. As conclusions, it is worth mentioning to say that students showed difficulties and obstacles at the beginning, due mainly to the verbal (spoken) or written comprehension of the problems, in the mathematical descriptions and in the mathematical communication. However, this fact is reverted with the methodology used, showing significant progresses at the end of the experience.

INTRODUCTION

As a result of the poor results obtained by students in the national test (SIMCE) and internationals (PISA y TIMSS), which depict that Chilean students have been much lower than the international media, and not showing significant differences in the last years. There is a permanent concern at all educational levels in order to improve the quality of the mathematical learning in Chile. To understand this problem thoroughly, it is necessary to analyze the mathematical formation in the last decades, since it has been aimed at exercising preferently and at the management of the mathematical operation and to the basic algorithmic outside contexts. This fact has not allowed the students to understand its usefulness in the present world. This problem has widely increased at schools that deal with medium to low socio economic strata, and even more, at the rural and marginal areas (Aravena & Caamaño, 2007).

Problem solving, with some exceptions, is far away from the classrooms of our country. It can be proved that mathematics is neither related to reality nor to the other areas of knowledge. Furthermore, there exists a division inside mathematics itself (Aravena & Caamaño, 2007).

There is a great amount of research that depicts the importance of problem solving in the development of the thinking and uppermost skills. Besides, different skills for the problem solving are shown as Heuristic methods that easy work for students.

¹ This present article is based on a wider research which has been directed by Dr. María Aravena Díaz in her Seminary to obtain a Postgraduate on Mathematical Education, which was developed by the following teachers of Basic Education: Carrión A. Zunilda; Garrido E. Fernando; Miño V. Elena; Muñoz B. Myriam; Fuentealba C. Eduardo; Morales G. Sandra.
Starting with Polya works (1957), which have been taken as a referent by most of the researches, we can find different proposals that evidence the importance that problem solving has in the formation of the present mathematics (Schoenfeld, 1982; Mayer, 1986; Polya, 1985; Schoenfeld, 1987; Schoenfeld, 1988; Santos, 1992; Shumizu, et al., 2007).

**Problem under Research**
Based on the above mentioned problematic, and on the researches made at an international level on problem solving. It was put forward a work proposal in order to introduce students of Primary Education in the problem solving method. For that purpose, the Lesson Study Model was introduced in the design of the lesson plan that aims at essential elements in the development of the mathematical reasoning, with the caution of adapting it to the socio-cultural Chilean reality, and to the 4 Polya stages. Which were introduced in the development of the class. Based on the exposed above, we put forward the following problem:
¿Which is the potential of work of students in Primary Education on solving problems with verbal and numeric enunciates using the 4 Polya stages?

**Objective of the Research.**
To validate a methodological proposal based on the Japanese model of Lesson Study, with the back up of George Polya problem solving method, proposing an improvement in the comprehension and problem solving with verbal or written and mathematical enunciates in mathematical and daily situations in students of Primary Education.

**General Hypothesis:**
The present results show that students of Chilean Primary Schools show difficulties and obstacles, which may be regulated using a detailed planning according to the Lesson Study Model and to the Polya model, as a methodological classroom strategy.

*Sub-hypothesis or particular hypothesis.*
H1: Students of the Chilean Primary Education evidence difficulties and obstacles when facing problems with verbal enunciates. For example: Reading Comprehension, organization and interpretation of data, mathematization (algorithms, properties, formulate regularities) and mathematical communication.

H2: Applying the Polya Method, based on problem solving, students improve their learning levels significantly, being able to develop comprehension skills, mathematical reasoning and mathematical communication.

**II. BACKGROUND.**
In different epochs and conditions, it has been proposed that doing mathematics is problem solving mostly. For ages, scientists have tried to understand and teach the necessary skills for that purpose. However, such history can be divided into two big moments: before Polya’s works (1945) and after these ones. In the first stage, Socrates first progresses outstand, especially in the construction of a square of equal area to the double of a given square, showing all the strategies and techniques of its solving process. Another support is observed in the work of Descartes where you can find suggestions for those who might like to solve problems easily. Besides, we can find significant help in Euler’s works, which are related to the techniques that he used and to the heuristic education he employed with his disciples.
The present stage is started with the works by Polya (1945), who impulses greatly to this process with his work “How to solve it”, and lately, with “Mathematical and Plausible Reasoning” (1954) y “Mathematical Discovery” (1965). Such works are a precious referent to the different groups that have worked and given support to this line. Since the 70’s, as a result of the crisis produced by “modern mathematics”, problem solving becomes the central axes of mathematics. (Schoenfeld, 1985), with the outstanding creation of Curricular Standards by the National Council of Teachers of Mathematics in the United States, that was assumed by several countries, becoming the fundamental objective in the Teaching of Mathematics and the axes of the curriculum in the 80’s. According to Rico (1988) and Brown (1983), this became the most important innovation, and also became an autonomous field of systematic research. Another important stimulus was done by the Cockcroft Report (1985), a comprehensive analysis of 11 mathematical themes, in England and Wales treats the main areas of difficulty, this has been referent for the research in this field. The 90’s stated a breakthrough in a systematic work with numerous publications, becoming an autonomous area within the Mathematical Education.

Dealing with the teaching of mathematics by problem solving as a subject of study, is becoming an outstanding issue. Besides, the different strategies of solving known as heuristic method, easy the work of students, and have been an international referent. But, in spite of all the proposals that aim at problem solving, the problematic has not still been solved in many countries and conditions, where they do not work with problems, and the Mathematics that is taught is completely far from context. As a result, students do not develop major skills they need, according to our present society.

**Problem Solving Model. Japanese Lesson Study.**

The results obtained by Japan in the International Tests of Mathematics, where it has got the first places, have caught the attention in our country. In an analysis of the text “Japanese Lesson Study in mathematics at a Glance” (Shizumi, et.al, 2005), the mathematical work in the classroom is focused on: 1) Teaching with the ‘Problem Solving Method’, (2) Teaching with the “Method of Discussion” and 3) Teaching with the ‘Method of Problem Discovery’. The problem solving method that roots on the theories developed by Dewey, Polya and Wallas, is the most used in the Japanese schools (Figure 1). The stages it is based upon are: (1) ’Comprehension of the problem’, (2) ‘Development of a solution by oneself’, (3) ‘Progress through the Discussion’, and (4) ‘Conclusion’. Each one of these steps is very well organized in the classes and is kept in the so called Annual Didactic Plan, according to national standards for the curriculum, and whose objective is to develop the comprehensive skill and creative reasoning. The following parts are taken into account: Connection of the previous content with the new one; development of the content and beginning of another one without any visible connection; subdivision of units to allow the spiral study.

The didactic plan of teaching, referred to as how to develop the classes aims at: (1) create situations or problems centred in the recognition of problems, in the recognition of regularities or properties that empower the inductive mathematical thinking and the search of a new information; (2) development of mathematical creative activities that stimulate the search of regularities; and (3) development of innovating teaching strategies in order to back up different ways of thinking and promoting the pleasure of learning. The purpose is to ensure that children learn by themselves through the stimulation of ideas about the necessary kind of knowledge for solving a problem. In the design of a Lesson Plan, the required didactic material is studied deeply, which acts as a
bridge or link for the students to develop their own ideas. Problems are prepared beforehand, going ahead to the ideas of the students, understanding the quality and efficiency of the ideas and developing questions in order to stimulate the solution (Aravena, 2007). In table 1, the three problem solving are shown upon the Japanese Model. We can find the 4 Polya. stages

<table>
<thead>
<tr>
<th>The 4 Polya stages</th>
<th>The 5 Dewey stages</th>
<th>The Wallas 4 stages</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Understanding of the problem</td>
<td>1. Experimenting a difficulty</td>
<td>1. Preparation</td>
</tr>
<tr>
<td>2. Sketching of an action plan</td>
<td>2. Defining the difficulty</td>
<td>2. Incubation</td>
</tr>
<tr>
<td>3. Execution of the plan</td>
<td>3. Generating a possible solution</td>
<td>3. Illumination</td>
</tr>
</tbody>
</table>

Table 1. Problem Solving Models based on the Japanese Method.

It can be inferred that the strategy for the developing of the class looks for empowering the methods, concepts, and forms of mathematical reasoning, which, are always aiming at all dimensions and levels and at the formulation and the solving of problems. It deals, then, with an integrating strategy in order to provide mathematics to the development of the mathematical reasoning, and also, to the communication of methods and processes. The strongest point of the Japanese teaching is the way how the teacher or group of teachers organizes the analysis of the planning, and the way they feedback the process, they call “Lesson Study”.

The three elements which are discussed in the study of a class are: (1) to design the lesson plan according to the national curriculum and to the principles that guide the method. This is, the formulation and discussion of the problems that will be worked by the students, focusing the classes towards the solving of such problems and the discussion of the didactic materials; (2) To develop the class work with the presence of the group of teachers; (3) To analyze the class with the team in order to detect the good points and the difficulties through the discussion and group reflection; y (4) to establish the adjustments for the feedback and the revision of the lesson plans. This modality, where teachers from different schools can be included has allowed them to reinforce and feedback the process and to establish improvements, as well as help other teachers to develop their own pedagogical practices.

III. METHODOLOGY

The methodology was qualitative and quantitative, a pretest-postest were designed, and a didactic unit that took into account the elements described in the model Lesson Study: (1) Mathematical problems that aim at: (a) development of the thinking (b) search for regularities and (c) solving of situations within the context of the students. (2) analysis of the required didactic material, studying such material deeply, so that it can act as a bridge or link between the students to develop their own ideas: (a) study of the problems and possible solutions on account of the teachers and (b) group revision of the possible difficulties and goals, discussing the quality and efficiency of the ideas, grounding and deepening the questions and possible answers.
Design of the Unit Planning. Problem Solving

Objective of the Unit: Solving problems from different areas of knowledge and from daily life that may involve the basic mathematical operations.

Didactic Material: Problems are prepared where regularities can be found and the mathematical background can be analyzed in each one of them, being solved by each one of the 6 teachers who took part in the experience, using methods and personal strategies. Afterwards, each one of the solving strategies is discussed by the team.

Students Behaviour: An analysis of the possible difficulties is made. The following aspects are considered: reading of the problem, conditions, restrictions, processes, types of solutions, previous knowledge, use of the didactic material.

Design of the lesson plan of each class. In the following Table an example of a class is presented.

<table>
<thead>
<tr>
<th>Class</th>
<th>Objective</th>
<th>Situation or problem</th>
<th>Planning of the students work</th>
</tr>
</thead>
<tbody>
<tr>
<td>Clase: (Nº 4 out of 6 classes.) of 90 minutes.</td>
<td>Understand The meaning of the concept and of the rest in order to apply it to daily situations.</td>
<td>Shelves For building shelves a carpenter needs the following: -4 long pieces of wood, -6 short pieces of wood, -12 small hooks, -2 big hooks, - 14 screws The carpenter has 26 long pieces of wood, 33 short pieces of wood, 200 small hooks, 20 big hooks, and 510 screws in the store. ¿How many complete shelves can this carpenter build? SKETCH FOR THE SHELF</td>
<td>a) Understanding of the problem:: (Time 15 minutes approximately) -Reading of the instructions and understanding of the situation. -Clearing of the problem situation through the discussion among students. b) Development of a solution by themselves: (Time 25 min) -The students think, discuss, and work on the problem and find solutions. -The teacher goes round the classroom commenting, orientating, making suggestions to the students who show difficulties to face the problem. c) Progress through discussion: (Time approx. 30 minutes) -A member of the group gives their solution to the class. -The students share ideas, exchange opinions about the qualities, advantages, and disadvantages of each solution, identifying likenesses and differences. d) Conclusion: (Time 15 minutes approx.) -Summary of the key points appeared in the class. -Consolidation of the ideas. -Record in the notebooks. e) Self evaluation or Auto evaluation (Time 5 minutes)</td>
</tr>
</tbody>
</table>

Table 2. Lesson plan 4 out of 6, taking the Japanese method Lesson Study

Methods and Instruments of Analysis
For recognizing the real progress of the work by the students, an interpretative analysis of the content was made. For that purpose, the following categories were designed beforehand: (1) **Understanding of the problem**: identifying data and conditions. (2) **Sketching of an action plan**: description of mathematical relationships that interpret the process. (3) **Execution of the plan**: discovering the regularities using the algorithms and properties. (4) **Verification of results**: Checking processes and checking results.
Interpreting the data and communicating the solutions (Polya, 1985; Aravena & Caamaño, 2007).

**Instruments of Analysis.** The didactic unit was validated through a triangulation of evaluators, selecting 10 problems to be incorporated in the classroom. The Pretest and posttest were validated through the Cronbach alfa, with the following results: pretest $\alpha = 0.9718$ y postest $\alpha = 0.9702$. Relative frequencies were used for the analysis of the data in each one of the categories. And, to give more validation to the study, the test student was used in a signification level of $0.05$. The items selected belonged to the indicators that are related to the 4 Polya stages, and that were analyzed through triangulation of the teachers and the researcher, and being sent to expert juries afterwards. Besides, a qualitative and quantitative analysis was made considering the categories under study and each one of the elements described in the model Lesson Study. For that purpose, an observation list was used, that was triangulated by the team.

**Sample.** The Maule Region is historically recognized as one of the most deficient in national tests measuring, especially in state schools of medium to low socio economic strata, and even more, at the rural and marginal areas as well, where you can find this diversity most clearly. (Fuentes, et. al, 1996; Aravena & Caamaño, 2007). For that reason, the Santa Rita School was chosen at Pelarco County, which is a rural area of the Maule Region. The work was done in students of 5th grade, ages ranging between 9 to 11, that was composed of students from the area from different family environments, from a medium socio economic stratum, coming from season workers families. The experience was implemented during a month during the second term of 2007, a group of 13 students.

**IV. DISCUSSION ON THE RESULTS**

The present text reports the results according to categories of analysis regarding to understanding of the problem of the verbal enunciated, the students show difficulties in identifying the data and the restrictions, the deficient results in this item reach to 62%, in contrast with the postest where it is observed that they make a correct identification 53.9%. Regarding this aspect, we coincide with the researches from Aravena & Caamaño (2007) that show that this deficiency continues to be present at secondary levels. Moreover, this situation continues to be present at Superior Education in Chile. (Aravena, 2002). Another difficulty is to identify the hidden or unknown mathematics, so that it can lead them to describe the mathematical relationships, the percentages of correct ones reaches only up to 40%. On the contrary, in the postest 53% identifies them as correct.

Regarding to a planning configuration in order to solve the problem, where data must be represented graphically, the percentages are inverted, since in the pretest the percentage was higher in this item, reaching to a 70% of the correct answers against a 46.2% in the postest. Regarding to discovering regularities, the results show that 100% of the students is not able to explain none of the regularities at the beginning, nor explaining the operating involved, as the multiplying principle. On the other hand, in the postest, the results are much encouraging, although they are not high, they are significant all the same, reaching to 46,2%.

Regarding to the indicator execution of a plan, the results show that, at the beginning, the students show serious problems about developing the algorithms, reaching only to 24% of the right ones. On the other hand, in the postest, the results are much more favourable, since 53% of the students execute them correctly. With regards to the application of the multiplying principle, 0% of the students does not reach it correctly,
against 38.5% in the posttest. Although the result is not the most favourable one, there is a meaningful change regarding to the pretest.

As to the verification of results, where we have wanted to include the written version of the answer to the real problem, the interpretation of the data communicating the solutions, the verification of processes and the proving of the results, so as to see if these match into the problem. The analysis of the results confirms that before the experience, all the items do not go beyond 40%. On the contrary, in the posttest the results surpass 53%. We coincide with Alsina (1998) y Aravena & Caamaño (2007) in this aspect. They show that the communication of results is one of the least worked out issues in teaching at all levels. So the students are not accustomed to giving an answer to the real problem, they just keep the mathematical problem.

QUALITATIVE DESCRIPTION

According to the categories of analysis, an interpretative analysis was done that was recorded by each one of the teachers. Through the afterthoughts and discussions inside the classroom, we detected that a planned work in the described terms was defined as:

(1) The role of the teacher will be to pay attention to the processes and help students to develop deductive reasoning skills, giving chance to students to express themselves, explain the way they reasoned, and that they can also be able to detect their weaknesses and mistakes.

(2) The students have an active role in the construction and development of the mathematical problems, where group work is fundamental for communicating ideas and the reinforcement of their self confidence in learning how to think by themselves, to defend and express their ideas to the rest.

(3) The classes came to be an accumulative product of dialogs, shown on the blackboard, where the students place the mathematical production to be analyzed by the class. It could be observed how they reason and think on the problem, the way the structure it, and how they enter into the mathematical problem without losing sight of the real problem.

(4) The reconstruction of the class that they developed in their notebooks becomes a potential source, since it reflects the evolving mathematical knowledge. The reproduction of the work carried out, leads them to rethink on how to reconstruct the class, allowing them to display significant notions, keeping an order in the content, deepening the understanding and thinking the learning process over. This allows to test the meta cognitive skill through the self regulation of the knowledge.
(5) The communication of concepts and mathematical processes was presented in a natural way, since the not penalizing of mistakes makes students express themselves freely.

(6) The evaluation was focused on recognizing what students are able to do instead of on what they are not. The auto evaluation or self evaluation that students did in each class allowed them to deepen in their own processes and regulate their learning process by themselves.

**ANALYSIS OF THE RESULTS THROUGH THE TEST T- STUDENT**

In order to provide more validation to the study, an analysis was done through the test t-student for individual samples. Equivalent items were taken in both tests and that were essential in problem solving according to the Polya four stages Model

<table>
<thead>
<tr>
<th>Name Items</th>
<th>Pretest Mean</th>
<th>SD</th>
<th>Posttest Mean</th>
<th>SD</th>
<th>Change Mean</th>
<th>SD</th>
<th>t-student</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Understanding the Problem</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Identifies data</td>
<td>1.92</td>
<td>1.32</td>
<td>2.85</td>
<td>1.068</td>
<td>0.92</td>
<td>0.862</td>
<td>**</td>
</tr>
<tr>
<td>Identifies unknowns quantity</td>
<td>2.00</td>
<td>1.21</td>
<td>2.92</td>
<td>1.038</td>
<td>0.92</td>
<td>0.954</td>
<td>**</td>
</tr>
<tr>
<td>Organizes details in writing</td>
<td>2.23</td>
<td>1.301</td>
<td>2.62</td>
<td>1.044</td>
<td>0.38</td>
<td>0.961</td>
<td>NS</td>
</tr>
<tr>
<td>Identifies operations</td>
<td>2.08</td>
<td>1.382</td>
<td>2.69</td>
<td>1.251</td>
<td>0.62</td>
<td>1.044</td>
<td>**</td>
</tr>
<tr>
<td><strong>Setting up a Plan</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Represents solution graphically</td>
<td>2.85</td>
<td>1.281</td>
<td>2.69</td>
<td>1.182</td>
<td>0.15</td>
<td>1.068</td>
<td>NS</td>
</tr>
<tr>
<td>Discovers regularities</td>
<td>0.92</td>
<td>0.277</td>
<td>2.23</td>
<td>1.363</td>
<td>1.31</td>
<td>1.581</td>
<td>***</td>
</tr>
<tr>
<td>Applies multiplicative principle</td>
<td>0.92</td>
<td>0.277</td>
<td>2.38</td>
<td>1.261</td>
<td>1.46</td>
<td>1.266</td>
<td>***</td>
</tr>
<tr>
<td><strong>Plan execution</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Solves algorithm division</td>
<td>1.38</td>
<td>1.261</td>
<td>2.77</td>
<td>1.481</td>
<td>1.38</td>
<td>1.557</td>
<td>NS</td>
</tr>
<tr>
<td>Solves algorithm multiplication</td>
<td>1.54</td>
<td>1.450</td>
<td>2.54</td>
<td>1.506</td>
<td>1.00</td>
<td>1.633</td>
<td>***</td>
</tr>
<tr>
<td>Expresses written regularity found</td>
<td>0.92</td>
<td>0.277</td>
<td>1.69</td>
<td>0.947</td>
<td>0.77</td>
<td>0.927</td>
<td>***</td>
</tr>
<tr>
<td><strong>Verification of results</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Expresses written response to the problem</td>
<td>1.85</td>
<td>1.405</td>
<td>2.62</td>
<td>1.121</td>
<td>0.77</td>
<td>1.166</td>
<td>**</td>
</tr>
<tr>
<td>Interprets and communicates data solutions</td>
<td>1.92</td>
<td>1.382</td>
<td>2.77</td>
<td>1.013</td>
<td>0.85</td>
<td>0.689</td>
<td>***</td>
</tr>
<tr>
<td>Checking of results</td>
<td>2.23</td>
<td>1.301</td>
<td>2.69</td>
<td>1.032</td>
<td>0.46</td>
<td>0.660</td>
<td>**</td>
</tr>
</tbody>
</table>

**Table 3. Results of the Test t-student considering the 4 Polya stages**

In table 3 we can observe that in 10 out of 13 items that were related to both tests, there are significant differences (p<0.05) and highly significant (p<0.01), favouring the postest. Thus, regarding to: (1) **Understanding of the problem**; it is observed that students show difficulties at the beginning in the identification of the unknown mathematics and in the restrictions of the problem. (2) **Sketching of an action plan**; to the solving regarding the solving, the description of the mathematical problem according to data problems. (3) **Regarding the plan execution**; It is established that students do not get to use the algorithms and their properties correctly. Also, it is very difficult for them to find the implicit regularities in the problems, especially when establishing a relationship between the mathematical data and the real problem. 4) **Verification of results**; where it is asked to communicate the solutions according to
the problem. We coincide with previous research that shows that this aspect is very little treated and worked out at all levels (Alsina, 1998). However, there are not significant differences regarding to the developing of schemes in order to organize data, especially in problems that are inside of a certain context. Based on the analysis shown, we can conclude that an organized planning in the above mentioned terms, allows students to use the problem solving skill for understanding the mathematical concepts and processes in a wider range. Thus, developing significant mathematical skills, what allows us to validate the sub hypothesis $H_1$ y $H_2$, and the hypothesis put forward in this research.

V. CONCLUSIONS AND DIDACTIC IMPLICATIONS
At a methodological level, it can be concluded that: (1) the planning designed let the students to have an active role in the solving of mathematical problems. (2) the group work let them develop self confidence in capacities and skills for learning and thinking by themselves.(3) an evaluation focused on recognizing what students are able to do instead of stressing the skills they are not able to get, can become a fundamental issue for students to lose their fear to make mistakes. (4) The use of the blackboard for students to expose their ideas allows the group to reconstruct their knowledge. (5) The importance of the role of the teacher for letting students express freely, considering mistakes as a natural thing. Finally, we can restate that an organization in the above mentioned terms is a promising one if we want to develop major skills. Lastly, it would be quite convenient to apply this experience in similar situations so as to validate this study, and focus it on new studies from the qualitative point of view, using case study to know how students reconstruct their mathematical knowledge deeply..

VI. REFERENCES.


POLYA (1954). “Mathematical and Plausible Reasoning”


An ICT environment to assess and support students’ mathematical problem-solving performance in non-routine puzzle-like word problems

Paper to present in TSG19 at ICME11
Theme (d): Research and development in problem solving with ICT technology

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Summary
This paper reports on a small-scale study on primary school students’ problem-solving performance. In the study, problem solving is understood as solving non-routine puzzle-like word problems. The problems require dealing simultaneously with multiple, interrelated variables. The study employed an ICT environment both as a tool to support students’ learning by offering them opportunities to produce solutions, experiment and reflect on solutions, and as a tool to monitor and assess the students’ problem solving processes. In the study, 24 fourth-graders were involved from two schools in the Netherlands. Half of the students who belonged to the experimental group worked in pairs in the ICT environment. The analysis of the students’ dialogues and actions provided us with a detailed picture of students’ problem solving and revealed some interesting processes, for example, the bouncing effect that means that the students first come with a correct solution and later give again an incorrect solution. The test data collected before and after this “treatment” did not offer us a sufficient basis to draw conclusions about the power of ICT environment to improve the students’ problem-solving performance.

1. INTRODUCTION

Problem solving is a major goal of mathematics education and an activity that can be seen as the essence of mathematical thinking (Halmos, 1980; NCTM, 2000). With problems tackled in problem solving typically defined as non-routine (Kantowski, 1977), it is not surprising that students tend to find mathematical problem solving challenging and that teachers have difficulties preparing students for it. Despite the growing body of research literature in the area (Lesh & Zawojewski, 2007, Lester & Kehle, 2003, Schoenfeld, 1985), there is still much that we do not know about how students attempt to tackle mathematical problems and how to support students in solving non-routine problems.

In order to get a better understanding of Dutch primary school students’ competences in mathematical problem solving, the POPO study started in 2004. In this study, 152 fourth-grade students who are high achievers in mathematics were administered a paper-and-pencil test on non-routine problem solving. In a few items, students were asked to show their solutions strategies. The results were disappointing. Students did not show a high performance in problem solving, despite their high mathematics ability (Van den Heuvel-Panhuizen, Bakker, Kolovou, & Elia, in preparation). Although the students’ scribbling on the scrap paper gave us important information about their solution strategies, we were left with questions about their solution processes. Moreover, after recognizing that even very able students have difficulties with solving the problems, we wondered what kind of learning environment could help students to improve their problem solving performance. The POPO study thus yielded a series of questions. To answer these questions we started the iPOPO study which – in accordance with the two main questions that emerged from the POPO study – implied a dual research goal.

First, the iPOPO study aimed at gaining a deeper understanding of the primary school students’ problem solving processes, and, second, it explored how their problem-solving skills can
be improved. For this dual goal of assessing and teaching, the study employed ICT both as a tool to support students’ learning by offering them opportunities to produce solutions, experiment and reflect on solutions, and as a tool to monitor and assess the students’ problem solving processes. In particular, we designed a dynamic applet called Hit the target, which is based on one of the paper-and-pencil items used in the POPO study. Like several of these items, it requires students to deal with multiple, interrelated variables simultaneously and thus prepares for algebraic thinking.

This paper focuses on the following two research questions: Which problem-solving strategies do fourth-grade students deploy in this Hit the target environment? Does this ICT environment support the students’ problem-solving performance?

2. THEORETICAL BACKGROUND

2.1. Mathematical problem solving

The term “problem solving” is used for solving a variety of mathematical problems, ranging from real-life problems to puzzle-like problems. Our focus is on the latter. We consider problem solving as a cognitive activity that entails strategic thinking, and that includes more than just carrying out calculations. An episode of problem solving may be considered as a small model of a learning process (D’Amore, & Zan, 1996). In problem solving, the solution process often requires several steps. First the students have to unravel the problem situation. Subsequently, they have to find a way to solve the problem by seeking patterns, trying out possibilities systematically, trying special cases, and so on. While doing this they have to coordinate relevant mathematical knowledge, organize the different steps to arrive at a solution and record their thinking. In sum, in our view problem solving is a complex activity that requires higher order thinking and goes beyond standard procedural skills (cf., Kantowski, 1977).

An example of a mathematical problem used in the POPO study is shown in Figure 1. Someone who knows elementary algebra might use this knowledge to find the answer to this problem by, for example, solving the equation 2x – 1(10 – x) = 8. Fourth-grade students, however, have not yet learned such techniques, but can still use other strategies such as systematic listing of possible solutions or trial and error. Grappling with such problems might be a worthwhile experiential base for learning algebra in secondary school (cf., Van Amerom, 2002).

Within the complexity that characterizes problem-solving activity, D’Amore and Zan (1996) identify the involvement of three interrelated discrete variables, as follows: the subject who solves the task; the task; and the environment in which the subject solves the task. This study primarily focuses on the third variable, referring to the conditions, which may help a subject to improve his problem solving abilities.
The research questions stated in Section 1 address two different aspects that are closely related: monitoring learning and supporting that learning. We have chosen to use ICT for both of these aspects, because – as Clements (1998) recognized – ICT (1) can provide students with an environment for doing mathematics and (2) can give the possibility of tracing the students’ work.

2.2. ICT as a tool for supporting mathematical problem solving

A considerable body of research literature has shown that computers can support children in developing higher-order mathematical thinking (Suppes, 1966; Papert, 1980; Clements & Meredith, 1993; Sfard & Leron, 1996; Clements, 2000; Clements, 2002). Logo programming, for example, is a rich environment that elicits reflection on mathematics and one’s own problem-solving (Clements, 2000). Suitable computer software can offer unique opportunities for learning through exploration and creative problem solving. It can also help students make the transition from arithmetic to algebraic reasoning, and emphasize conceptual thinking and problem solving. According to the Principles and Standards of the National Council of Teachers of Mathematics (NCTM, 2000) technology supports decision-making, reflection, reasoning and problem solving.

Among the unique contributions of computers is that they also provide students with an environment for testing their ideas and giving them feedback (Clements, 2000). In fact, feedback is crucial for learning and technology can supply this feedback (NCTM, 2000). Computer-assisted feedback is one of the most effective forms of feedback because “it helps students in building cues and information regarding erroneous hypotheses”; thus it can “lead to the development of more effective and efficient strategies for processing and understanding” (Hattie & Timperley, 2007, p.102). More generally, computer-based applications can have significant effects on what children learn because of “the computer’s capacity for simulation, dynamically linked notations, and interactivity” (Rochelle, Pea, Hoadley, Gordin, & Means, 2000, p. 86).

This learning effect can be enhanced by peer interaction. Pair and group work with computer software can make students more skilful at solving problems, because they are stimulated to articulate and explain their strategies and solutions (Wegerif & Dawes, 2004). Provided there is a classroom culture in which students are willing to provide explanations, justifications, and arguments to each other, we can expect better learning.

2.3. ICT as a window onto students’ problem solving

Several researchers have emphasized that technology-rich environments allow us to track the processes students use in problem-solving (Bennet & Persky, 2002). ICT can provide mirrors to mathematical thinking (Clements, 2000) and can offer a window onto mathematical meaning under construction (Hoyles & Noss, 2003, p. 325). The potential of computer environments to provide insight into students’ cognitive processes makes them a fruitful setting for research on how this learning takes place.

Because software enables us to record every command students make within an ICT environment, such registration software allows us to assess their problem solving strategies in more precise ways than can paper-and-pencil tasks. Therefore, computer-based tasks as opposed to conventional paper-and-pencil means have received growing interest in the research literature for the purposes of better assessment (Clements 1998; Pellegrino, Chudowsky, & Glaser, 2001; Bennet & Persky, 2002; Burkhardt & Pead, 2003; Threlfall, Pool, Homer, & Swinnerton, 2007; Van den Heuvel-Panhuizen, 2007).

Where early-generation software just mimicked the paper-and-pencil tasks, recent research shows that suitable tasks in rich ICT environments can also bring about higher-order problem solving. For example, Bennet and Persky (2002) claimed that technology-rich environments tap important emerging skills. They offer us the opportunity to describe performance with something more than a single summary score. Furthermore, a series of studies indicated that the use of ICT facilitates the assessment of creative and critical thinking by providing rich environments for problem solving (Harlen & Deakin Crick, 2003).

By stimulating peer interaction we also expect that students will articulate more clearly their thinking than when working individually. Thus, student collaboration has a twofold role: it helps them shape and broaden their mathematical understandings and it offers researchers and teachers a nicely bounded setting in order to observe collaboration and peer interaction (Mercer & Littleton, 2007).
3. METHOD

3.1. Research design and subjects
The part of the iPOPO study described in this paper is a small-scale quasi-experiment following a pre-test-post-test control group design. In total, 24 fourth-graders from two schools in Utrecht participated in the study. In each school, 12 students who belonged to the A level according the Mid Grade 4 CITO test – in other words to the 25% best students according to a national norm – were involved. Actually, the range of the scores that correspond to level A of the Mid Grade 4 CITO test is between 102 and 151 points. In both schools, the average mathematics CITO score of the class was A and the average “formation weight” of the class and the school was 1. This means that the students were of Dutch parentage and came from families in which the parents had at least secondary education. First, of each school six students were selected for the experimental group. Later on, the group of students was extended with six students to be in the control group. These students also belonged to the A level, but, unfortunately, their average score was lower than that of the experimental group. The teacher obviously selected the more able students first.

An ICT environment was especially developed for this study to function as a treatment for the experimental group. Before and after the treatment, a test was administered as pre-test and post-test. The control group did the test also two times, but did not get the treatment in between. The quasi-experiment was carried out in March-April 2008. The complete experiment took about four weeks: in the first week the students did the test, in the second week the experimental group worked in the ICT environment and in the fourth week the students did again the test.

3.2. Pre-test and post-test
The test that was used as pre-test and post-test was a paper-and-pencil test consisting of three non-routine puzzle-like word problems, titled Quiz (see Figure 1), Ages, and Coins. The problems are of the same type and require that the students deal with interrelated variables. The test sheets contain a work area on which the students had to show how they found the answers. The students’ responses were coded according to a framework that was developed in our earlier POPO study. The framework covers different response characteristics including whether the students gave specific strategy information, how they represented that strategy and what kind of problem-solving strategies they applied.

3.3. Applet used as treatment
The treatment consisted of a Java applet called Hit the target. It is a simulation of an arrow shooting game. The screen shows a target board, a score board featuring the number of gained points, and the number of hit and missed arrows, a frame that contains the rules for gaining or loosing points, and an area in which the number of arrows to be shot can be filled in. A hit means that the arrow hits the yellow circle in the middle of the target board; then the arrow becomes green. A miss means that the arrow hits the gray area of the board; in that case, the arrow becomes red.

![Figure 2: Screen view of applet in the computer-shooting mode](image)

1 The applet has been programmed by Huub Nilwik.
The applet has two modes of shooting: a player shoots arrows by him or herself or lets the computer do the shooting (see Figure 2). In case the player shoots, he or she has to drag the arrows to the bow and then draw and unbend the bow. The computer can do the shooting if the player selects the computer-shooting mode and fills in the number of arrows to be shot. Regarding the rules for gaining points there are also two modes: the player determines the rules or the computer does this. The maximum number of arrows is 150 and the maximum number of points the player can get by one shot is 1000.

As the player shoots arrows or lets the computer do so, the total score on the scoreboard changes according to the number of arrows shot and the rules of the game. The player can actually see on the scoreboard how the score and the number of hits and misses change during the shooting. The player can also remove arrows from the target board, which is again followed by a change in the total score. When the player wants to start a new shooting round, he or she must click on the reset button. The player can change the shooting mode or the rules of the game at any time during the game.

The aim of the applet is that the students obtain experience in working with variables and realize that the variables are interrelated (see Figure 3); a change in one variable affects the other variables. For example, if the rules of the game are changed, then the number of arrows should be also changed to keep the total points constant.

The 12 students of the experimental group worked for about 30 minutes in pairs with the applet. The pairs were chosen by the researcher in such a way that all of them would have about the same average CITO score and consisted of a boy and a girl. The dialogue between the students and their actions on the applet were recorded by Camtasia software, which captures the screen views and the sound in a video file. Scrap paper was also available to the students. Before the students started working, it was explained to them that they should work together, use the mouse in turns, explain their thinking to each other, and justify their ideas.

The work with the applet started with five minutes of free playing in which the students could explore the applet. Then, they had to follow a pre-defined scenario containing a number of directed activities and three questions (see Table 1). The first two questions (A and B) are about the arrows while the rules of the game and the gained points are known. In the third question (C), which consists of two parts, the rules of the game are unknown.

Table 1: Questions in the pre-defined scenario

<table>
<thead>
<tr>
<th>Arrows</th>
<th>Rules</th>
<th>Gained points</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>A. How many hits and misses?</strong></td>
<td>Hit +3 points, miss –1 point</td>
<td>15 points</td>
</tr>
<tr>
<td><strong>B. How many hits and misses?</strong></td>
<td>Hit +3 points, miss +1 point</td>
<td>15 points</td>
</tr>
<tr>
<td>15 hits and 15 misses</td>
<td><strong>C1. What are the rules?</strong></td>
<td>15 points</td>
</tr>
<tr>
<td></td>
<td><strong>C2. Are other rules possible to get the result 15 hits-15 misses-15 points?</strong></td>
<td></td>
</tr>
</tbody>
</table>
one arrow, followed by shooting two arrows and then a few more, in order to get five arrows on
the target board. Their attention was then drawn to the scoreboard; they had five hits and zero
misses and their total score was zero since the rules of the game had been initially set to zero.
After that, the rules were changed so that a hit meant that three points were added. Then, the
students had to shoot again five arrows in both shooting modes, each resulting in a total score of
15 points. Afterward, the rule was changed again. A miss then meant that one point had to be
subtracted. At this point, Question A was asked, followed by Questions B and C.

4. RESULTS

4.1. The students’ problem-solving strategies in the ICT environment
All pairs were successful in answering the Questions A, B, and C. The solutions were found based
on discussions and sharing ideas for solutions. In all cases, explanations were provided and the
talk between the students stimulated the generation of hypotheses and solutions. However, some
students provided more elaborate explanations and suggested more successful problem-solving
strategies than others.

In order to identify the problem-solving strategies the students applied, we analyzed all
dialogues between the students. In this paper, however, we will only discuss our findings with
respect to Questions C1 and C2, which triggered the richest dialogues.

Characteristic for Question C is that the number of hits and misses, and the number of points
were given, but that the students had to find the rules. All pairs were able to answer Questions C1
and C2, and most of them could generalize to all possible solutions (“It is always possible if you do
one less”), albeit on different levels of generalization. The Tables 2 and 3 show which strategies the
pairs used when solving Questions C1 and C2. Each pair of students is denoted with a Roman
numeral. Pairs I, II, and III belong to school A, while Pairs IV, V, and VI belong to school B.

Table 2: Problem-solving strategies when solving C1

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Pairs</th>
<th>Average CITO score per pair</th>
</tr>
</thead>
<tbody>
<tr>
<td>1a</td>
<td>Directly testing a correct solution (+2 –1 or +1 +0)</td>
<td>I* 1 1 1 1</td>
</tr>
<tr>
<td>2a</td>
<td>Testing incorrect canceling-out solution (+1 –1)</td>
<td>1</td>
</tr>
<tr>
<td>2b</td>
<td>Testing other incorrect solution(s)</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>Adapting the rules of the game until a correct solution is reached</td>
<td>2 2</td>
</tr>
</tbody>
</table>

Number of trials

1 1 1 2 1 3

* The numbers in the cell indicate the order in which the strategies were applied

When answering Question C1 (see Table 2), four out of the six pairs directly came up with a correct
solution. Pair VI found the correct solution in the third trial. The most interesting strategy came from
Pair IV. This pair found the correct solution in the second trial. The pair started with a canceling-out
solution (+1 –1) resulting in a total score of zero and then changed the solution to get 15 points in
total.

Table 3 shows that having found a correct solution in C1 did not mean that the students had
discovered the general principle (or the correct solution rule) of getting “15 hits-15 misses-15
points”. Even after finding the correct solution rule and generating a series of correct solutions,
some students tested wrong solutions again (we could call this the “bouncing effect”). Perhaps
they were not aware that there is only one correct solution rule; the difference between the number
of points added for every hit and the number of points subtracted for every miss (or vice versa)
should be 1, or the difference between the number of hit-points and miss-points should be 15
points. The highest level of solution was demonstrated by Pair VI, who recognized that the
difference between the points added and the points subtracted should be 15 (and that explains why
the difference between the number of points added for every hit and the number of points
subtracted for every miss – or vice versa – should be 1). A clever mathematical solution came
from the Pairs I and II. These students just used the correct solution to C1 in the reverse way to get
the required result of 15 points in total.
<table>
<thead>
<tr>
<th>Strategy</th>
<th>Pairs</th>
<th>Average CITO score per pair</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>I</strong></td>
<td><strong>II</strong></td>
<td><strong>III</strong></td>
</tr>
<tr>
<td>4a Repeating the correct solution to C1</td>
<td>111</td>
<td>111</td>
</tr>
<tr>
<td>4b Reversing the correct solution to C1 to find another correct solution ((-1 +2) or (-0 +1/0 +1))</td>
<td>1*</td>
<td>1/3</td>
</tr>
<tr>
<td>5a Generating a correct solution rule based on testing of (a) correct solution(s) for which the difference between the number of points added for every hit and the number of points subtracted for every miss (or vice versa) is 1</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>5b Generating a correct solution rule based on understanding that the difference between hit-points and miss-points is 15</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>5c Generating a general correct solution rule (“the difference of 1 also applies to 16-16-16”)</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td>6 Testing more correct solutions from a correct solution rule</td>
<td>3</td>
<td>7</td>
</tr>
<tr>
<td>2b Testing other incorrect solution(s)</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>7 Generating an incorrect solution rule (keeping ratio 2:1 or using rule +even number –odd number) based on correct solution(s)</td>
<td>2/4</td>
<td></td>
</tr>
</tbody>
</table>

* The numbers in the cell indicate the order in which the strategies were applied

Besides strategies that directly or indirectly lead to a correct solution or rule, some other characteristics were found in the solution processes (see Table 4). Four pairs altered or ignored information given in the problem description. It is noteworthy that during subsequent attempts to answer Question C2, some students insisted on keeping the rules constant and changing the number of hits and misses in order to get a total of 15 points. Pair V, for example, changed the problem information (15 hits and 15 misses) and started C2 with trying out the solution 1 hit is 15 point added and 1 miss is 15 points subtracted. The total score then became zero; subsequently, they set the number of hits to 30 and the number of misses to 15, which resulted into a high score. Even though at that point the researcher repeated the correct problem information, the students ignored it persistently. In their third attempt, they changed the number of hits into 1 and 0 respectively and the total score became 15 instead of the reverse (15 hits and 15 misses resulting in 15 points). Only when the researcher repeated the question they considered the correct information and tried out the solution +4 –2 with 15 hits and 15 misses. However, the total score was 30 points and they suggested doubling the number of misses to 30 so that the number of total points would be halved. This is clearly an example of a wrong adaptation. Another example is from Pair VI. After having +3 and –1 as the rule of the game, resulting in a total of 30 points, the students change the number of hits into 10 in order to get 15 points as the result but forgetting that the number of hits should be 15.

<table>
<thead>
<tr>
<th>Characteristics</th>
<th>C1</th>
<th>C2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Altering or ignoring information</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>Exploring large numbers ((\geq 1000))</td>
<td>X</td>
<td>X</td>
</tr>
</tbody>
</table>

Another characteristic of the solution processes was testing rules including large numbers. Four of the six pairs tried out numbers bigger than 1000. These explorations all took place when answeringProbleme-solving strategies when solving C2

**Table 3: Problem-solving strategies when solving C2**

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Pairs</th>
<th>Average CITO score per pair</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>I</strong></td>
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</tr>
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<td>1*</td>
<td>1/3</td>
</tr>
<tr>
<td>5a Generating a correct solution rule based on testing of (a) correct solution(s) for which the difference between the number of points added for every hit and the number of points subtracted for every miss (or vice versa) is 1</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>5b Generating a correct solution rule based on understanding that the difference between hit-points and miss-points is 15</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>5c Generating a general correct solution rule (“the difference of 1 also applies to 16-16-16”)</td>
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<td></td>
</tr>
<tr>
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<td>7</td>
</tr>
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<td>2</td>
</tr>
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<td>2/4</td>
<td></td>
</tr>
</tbody>
</table>

* The numbers in the cell indicate the order in which the strategies were applied
the second part of Question C. The students found this working with large numbers quite amusing, since they then could get a large amount of total points. That the students worked with numbers larger than 1000 was quite remarkable, because it was not possible to fill in numbers of this size in the applet. Consequently, the students had to work out the results mentally. It is also worth noting that some students understood that one could go on until one million or one trillion (Pair IV). This means that several students knew that there are infinite solutions, as it was made explicit by one pair (see Pair II). Furthermore, most of the students used whole numbers and no one used negative numbers. In one occasion, a student (from Pair II) suggested adding 1½ points for a hit, but the applet does not have the possibility to test solutions with fractions or decimals.

Observing the students while working on the applet revealed that the students demonstrated different levels of problem-solving activity. For example, there were students that checked the correctness of their hypotheses by mental calculation, while others just tried out rules with the help of the applet. None of them questioned the infallibility of the applet; when they used the applet after they had found out that a rule was wrong, they did this to make sure that they were really wrong. Furthermore, the students also showed differences in the more or less general way in which they expressed their findings. One of the students articulated that the general rule “a hit is one point more (added) than the number of points (subtracted) by a miss” also applies to other triads such as 16 hits-16 misses-16 points and in general to all triads of equal numbers.

To conclude this section about the ICT environment, we must say that observing the students while working with the applet gave us quite a good opportunity to get closer to the students’ problem-solving processes.

4.2. Does the ICT environment support the students’ problem-solving performance?

In this section, we discuss the results from the pre-test and the post-test in the experimental and control group. Figure 4 shows the average number of correct answers per student in both groups in school A and school B.

Figure 4: Average number of correct answers per student in the pre and the post-test in both groups

As can be seen in Figure 4, if the group of students is taken as a whole, the experimental group gained slightly from the treatment. However, we have too few data to give a reliable answer to the research question. Only 12 students from school A and 12 students from school B were involved in this study and among these schools, the results were quite different. Only in school A, there is a
considerable improvement in the scores of the post-test. Another issue is the mismatch between experimental and control group (see also Section 3.1). In both schools, the control group scored lower than the experimental group. This mismatch was more evident in school A. A plausible explanation for these differences could be that although all students had an A score in mathematics, the average CITO scores of the experimental group and the control group were different in school A and school B (see Figure 5).

In fact, the differences between the average CITO score of the experimental and control group in each school, presented in Figure 5, are similar to the differences between the average scores of these groups in the paper-and-pencil test. In school A, the control group has a lower CITO score than the experimental group. The same holds for school B, but there the difference is smaller than in school A.

5. DISCUSSION

We started this study with two questions that emerged from the earlier POPO study. To investigate these questions, we set up, as a start, a small-scale study in which an ICT environment played a crucial role. The dialogues between the students and their actions when working in the ICT environment gave us a first answer to the first research question. The collected data provided us with a detailed picture of students’ problem solving and revealed some interesting processes, for example, the bouncing effect and the making of wrong adaptations.

Our second question is difficult to answer. The sample size, and the number of the test items were not sufficient to get reliable results and the time we had at our disposal was not enough to gather and analyze more data. Moreover, the time that the experimental group worked in the ICT environment was rather limited to expect an effect. Despite these shortcomings, we decided to carry out a small-scale study in order to try out the test items and the ICT environment with a small group of students first.

Clearly, more data (more students, more schools and more problems) are needed to confirm or reject our conjecture that having experience with interrelated variables in a dynamic, interactive ICT environment leads to an improvement in problem solving performance. For this reason, we will extend our study to larger groups of students, involving students of higher grades and different mathematical ability levels as well. Moreover, to see more of an effect we will enlarge the working in the ICT environment substantially. In addition, we will extend the study by analyzing the students’ problem-solving strategies when solving paper-and-pencil problems. Our experiences from the present study will serve as a basis for doing this future research.

REFERENCES


THE COMPUTER TOOL FOR VERIFICATION HYPOTHESES IN PARAMETRICAL PROBLEMS SOLVING

D. Mantserov, D. Petrichenko, S. Pozdnyakov

Summary

The article considers various aspects of using verification environment (so-called «Verifier») to support students’ activity in parametrical problems solving. The Verifier was created to support high school students in their work with functions. It compares student’s answer with true one and shows her counterexamples accompanied by their graphs and comments. So with Verifier’s support one may improve her own solution applying various conjectures. A student should consider various conditions and different cases; therefore the Verifier’s answers have to be given in a complex logical form, which was a difficulty we got over in our work. We believe that our approach opens the way for expanding types of problems, which will be useful for studying function properties.

The study and research questions. Relevant features of the theoretical framework

Problem solution is important part of mathematics teaching methodology. In the process of finding a solution to the problem some features of creative thinking dynamics are shown (Wertheimer, [1]):
- Collision with a problem;
- Ambiguity, incompleteness of a situation;
- Refinement of infringement areas;
- Using operations to change situation.

The solving process also includes conjectures making and partial solution construction (see, e.g. Polyga [3, 4]).

The heuristics play an important role in mathematical problem solving (see, e.g. Schoenfeld [2]). It is interesting to consider possibilities for technological support of heuristic activities. For doing this we should base on the
- essence of didactic task;
- psychology of intellectual activity;
- computer tools usage traits or patterns;

The main question for our research is:

How can we support productive activity of students in solving mathematical problems with logically complex solutions via computer tools usage?

To find the answer we will explore an idea of providing a student with examples and counterexamples during the solution process.

We will use M. Minsky conception [5] of two mind mechanisms for combining separate facts in new entity:
1) creating new abstraction for these facts;
2) mechanical combining of facts in new collection.

In our research we will use both mechanisms: the first one for human-computer interaction, and the second one for organizing computer support.
In problem solving process the students should construct their answers in the predicate form, so that the Verifier could check them against the set of existing examples. But one may ask why should they be of such importance for a student? First of all, it is so because abstract conceptions in human mind are closely connected with their examples ("by default") [5] and it is possible to form the new concepts only through considering “pros” and “contras” [6].

The graphical interface of Verifier is based on classical teaching routine - the "IRE sequence", which consists of three steps: a teacher initiates an interaction, a student responds then, and the teacher evaluates the response (see, e.g. Sinclair & Coulthard, 1975 [7], Cazden, 1988 [8]).

Design and methodology

Object classes and their properties description

Verifier’s tasks are based on parametric classes for functions such as $y=\text{kx+b}$, $y=\text{alx-b l + l x-c l}$, $y=\text{cx}^{m/n}$, $y=\log_a(\text{kx+b})$, $y = a^x + \text{kb}^x$ (Verifier’s environment allows to construct new classes). Every class is connected with the set of examples associated with parameters' values - which are interpreted by the Verifier to show function formulae and graphs.

Each task related to a function class may be interpreted as the class property.

Let's consider a class typical for studying in 10-11 grades of high school in Russia. It is the class of logarithmic functions which is determined by the formula $y=\log_a(\text{bx+c})$ and a predicate which restricts the domain of parameters, for example, $a>0 \& a \neq 1$.

Various properties can be formulated as predicates of class parameters $a$, $b$, $c$.

Examples.

1) “$y=\log_a(\text{bx+c})$ is increasing function”: $(a>1 \& b>0)$ OR $(a<1 \& b<0)$.
2) “A domain of $y=\log_a(\text{bx+c})$ is all positive numbers”: $b>0 \& c=0$.
3) “An intersection of function graph with y-axe is lying in upper semiplane”: $(a>1 \& c>1)$ OR $(a<1 \& c<1 \& c>0)$.
4) “Function graph neither has points inside the III quarter of coordinate plane nor on the quarter boundaries”: $(a<1 \& b>0 \& c<1 \& c>0)$ OR $(a>1 \& b<0 \& c>1)$.

One may note that the last condition may be rewritten in other way:

$((a<1 \& b>0) \ OR \ (a>1 \& b<0)) \ & \ ((a>1 \ & c>1) \ OR \ (a<1 \ & c<1 \ & c>0))$,

It means that this property can be expressed through previous ones:

“Function graph neither has points inside the III quarter of coordinate plane nor on the quarter boundaries” can be written as

($y=\log_a(\text{bx+c})$ is increasing function”) & (“Function graph neither has points inside the III quarter of coordinate plane nor on the quarter boundaries”)
The last example shows that after describing some set of properties via predicates we can verify the connections between these properties.

**Examples of interactive answer detailing for Verifier problems**

Let’s consider two basic examples and demonstrate various logical paths while solving problems with Verifier.

**Problem.** Find conditions for fraction power function $y=cx^{m/n}$ with the range of all positive real numbers ($m$ and $n$ are coprime numbers, $n>0$).

**First trial**

*Presumable student reasonings.* «We know that fraction power function $y=x^{m/n}$ has only positive values, therefore $c$ must be positive».

**Input:** $c>0$

**Output.** Your answer has a mistake. Look at the picture. This function satisfies your condition but doesn't satisfy the task condition.

**Judgment:** "We need to narrow fraction power function range. So we must narrow power function’s domain. To narrow fraction power function’s domain we need to make it integer power function”.

---

**Second trial**

**Input:** $c>0 \& n>1$

**Output.** Your answer has a mistake. Look at the picture. This function satisfies your condition but doesn't satisfy the task condition.

**Judgment:** «Yes, of course. The function must not be defined for $x=0$ therefore $m$ must be negative».

---

**Third trial**

**Input:** $c>0 \& n>1 \& m<0$

**Output.** Your answer has a mistake. Look at the picture. This function satisfies task condition but doesn't satisfy your condition.

**Judgment:** "Really. Narrowing fraction power function domain can be done without narrowing the function domain. The function take the same value twice. Let's return to our hypothesis of integer power functions”.

---
Fourth trial

Input: $c > 0 \& m < 0 \& n \geq 1$

Output. Your answer has a mistake. Look at the picture. This function satisfies your condition but doesn't satisfy the task condition.

Judgment: "It is clear the function must be odd. Let's try to check the idea by taking $m = -2$.

Fifth trial

Input: $c > 0 \& m < 0 \& n \geq 1 \& m = -2$

Output. Your answer has a mistake. Look at the picture. This function satisfies task condition but doesn't satisfy your condition.

Judgment: "Yes the conjecture proves to be true. To make function even we must take even $m$.

Sixth trial

Input: $c > 0 \& m < 0 \& n \geq 1 \& m \_i\_s\_e\_v\_e\_n$

Output. Your answer has a mistake. Look at the picture. This function satisfies task condition but doesn't satisfy your condition.

Judgment: "Ok. It appears that first idea works for functions with domain of all positive numbers, and the second one works for functions with domain of all numbers except zero. We must combine both ideas in a more accurate way".

Seventh trial

Input: $c > 0 \& m < 0 \& (n > 1 \mid n = 1 \& m \_i\_s\_e\_v\_e\_n)$

Output. Congratulations! It is not possible to find difference between your answer and true one using accessible examples.

Let's consider another way of deduction while solving the same problem (it was proposed by a student).
Problem. Find conditions for fraction power function \( y = cx^{m/n} \) to have range of all positive real numbers (\( m \) and \( n \) are coprime numbers, \( n > 0 \)).

First trial

Presumable student reasonings. «We know that fraction power function \( y = x^{m/n} \) has only positive values therefore \( c \) must be positive».

Input: \( c > 0 \)

Output. Your answer has a mistake. Look at the picture. This function satisfies your condition but doesn’t satisfy the task condition (here we see another counterexample).

Judgment: "Why this example doesn’t satisfy the task condition? The range does contain zero! How to eliminate it? We need to do power parameter negative!".

Second trial

Input: \( c > 0 \) & \( m < 0 \)

Output. Your answer has a mistake. Look at the picture. This function satisfies your condition but doesn’t satisfy the task condition (here we see another counterexample).

Judgment: "It is not enough to eliminate zero value from the range! We must eliminate all negative numbers too! Therefore the power parameter must be even".

Third trial

Input: \( c > 0 \) & \( m < 0 \) & \( m \text{ is even} \)

Output. Your answer has a mistake. Look at the picture. This function satisfies task condition but doesn’t satisfy your condition.

Judgment: "I forgot that the power parameter can be a fraction! We need to separate integer power parameters and fraction power parameters".

Fourth trial

Input: \( c > 0 \) & (\( m < 0 \) & \( n > 1 \) | \( n = 1 \) & \( m < 0 \) & \( m \text{ is even} \))

Output. Congratulations! It is not possible to find difference between your answer and true one using accessible examples.
Using Verifier to support problem reformulation

As M. Minsky noted [5], the reformulation of problem is one of the most important intellectual tools for its solving. For our purposes we will consider the process of reformulation as a process of knowledge representation forms alteration (Pozdnyakov, 1995, [9, 10]). For article's subject we will use four basic forms of knowledge representation:
- algorithmic representation (formulae)
- predicative representation
- representation by objects properties
- representation by examples collection.

So there are many types of transitions from one form of representation to another.

The partial list of such transitions:
- from formula to graph
- from graph and formula to properties
- from the properties of a “class” object to a concrete example of this object with such properties
- from some set of properties to new one
- etc.

Beneath we will provide for analysis of some types of problems based on transitions from one form of knowledge representation to another.

1. Type “from function graph and formula to function properties”

The type is characterized by giving the full information to a student about the object under consideration (function in our case). We give the set of examples and counterexamples of functions to a student with their formulae and graphs. The problem is to find an appropriate classification by formulating the property which separates examples and counterexamples.

2. Type “from function graph to function properties”

In this type of tasks we narrow the information about an object and provide the students with function graphs without any formulae.

The problem for student is to find an appropriate classification by formulating the property to separate examples and counterexamples based on function graphs comparing.
Student must be aware of function graphs properties to succeed in solving this type of problems.

3. Type “from function graph and formula to predicate”

By stating these problems we may form the skills needed for solving inequalities. A student can search for a predicate (it is the same as to describe a solution by a set of inequalities) to specify a set of objects. Formulae here play the role of additional help in solution search.

4. Type “from function formula to predicate”

Such type of the problems is characterized by function representation in a formula form. The search of solution requires a comparison between concrete formula and its class formula. This type of problems helps to train the skills of generating concrete examples for a common class and formulate characteristic properties of the object set.

5. Type “from function graph to predicate”

This is most important type of problems which combine common skills for information translation from one form to another with concrete math skills of exploring functions properties, solving equations and inequalities.

6. Type “from function formula to function properties”

The problems of such type are closely connected with translation of information from formulae language to verbal form. By such type problems we form math skills of object's "recognizing". For example: "what is an essential feature of function formula (from class given) to be even" or "what can be said about its domain."
Further implications

There are other types of problems, so here we provide their non-exhaustive list. They imply the examples construction in a symbolic or graph form.

7. Type “from predicate to function graph”.

8. Type “from predicate to function formula”.

9. Type “from predicate to function graph and formula”.

10. Type “from function properties to function graph and formula”.

Research results and discussion

Teachers’ answers on research questions

Beneath we cite some typical teachers’ answers which applied Verifier in their practice.

1. Is the material aimed to achieve new educational results?
"Yes. Students have learned to read attentively the problem description and to analyze it, noticing that problems, which seem to be identical, are different in fact".

2. Is it possible that Verifier usage in educational process have raised the level of student’s educational independence and self-activity?
"Yes, it is possible. The Verifier's problems put the students in a situation of self-control and self-estimations of actions".

3. Is it possible to state that using Verifier in educational process motivates students to learn math?
“Yes it is. The Verifier's problems are rather difficult and "time expensive", nevertheless students solved plenty of them, while working at home”.

4. What type of changes in real educational process was brought with the use of Verifier and were these changes necessary and effective?
“We didn’t feel any need in such changes”.

5. Does Verifier promote group and individual research activity in educational process?
“Yes. Verifier promotes different forms of group and individual research work”.

6. Does Verifier minimize labor expenses of the teachers when they prepare materials for lessons?
“Yes”.

7. Is Verifier suitable for schools, assuming it fits the hardware available at schools?

“Yes”.

8. What impact does Verifier have on quality of knowledge and skills?

“Students began to solve equations and inequalities with parameters more confidently. They also started to show awareness of function graphs properties”.

**Conclusion**

The results of experiments with Verifier for supporting problems solving approve the authors’ position that verification environment can be used by a student as a tool to support his thinking about the complex calculus problems.

This conclusion is in good correspondence to works of L. Vygotsky in psychology about the role of tools in intellectual skills forming [11].

**References**

THE DECISION-MAKING AS A SCHOOL ACTIVITY
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IES Luis Cobiella Cuevas (aantegue@yahoo.es)

ABSTRACT
A problem is chosen from the PISA 2003 Report, and then extended to include questions involving a choice of preferences and decisions. These questions are posed to secondary school students aged 15 to 16, whose problem-solving and heuristic decision-making skills are then analyzed. We discovered that students have difficulty using mathematics when assigning a weight to achieve an objective, though they are able to recognize the functional use of the rules of choice used in society.

INTRODUCTION
Some of the real world choices an individual may have to make require the use of mathematical calculation tools in order to establish a preference or analyze a situation. Multi-criteria analysis constitutes a way to model decision processes and involves the decision to be made, any unknown events which may affect the results, the possible courses of action and the result(s) of said actions. Using multi-criteria models, the decision maker assesses the possible implications each course of action may entail so as to obtain a better understanding of the relationships between the actions and the objectives. There are various mathematical procedures for summarizing the values yielded by each alternative with respect to all the criteria considered in the analysis. The best known mechanisms are those which use linear weighting (scoring), i.e. the simple sum of each attribute’s contributions. This is a very fruitful field of Operational Research in which improvements and variants are constantly being developed, along with growing applications in various contexts (Berumen and Llamazares, 2007).

For the specific case of a discrete decision problem, the information available is realized through an evaluation of n attributes for a set of m alternatives, and presented in a double entry matrix in which the attributes are shown as columns, while the associated alternatives appear in the rows. The decision is made by fixing the criterion to be used in deciding the best solution (Romero, 1993).

In “The Best car” activity in PISA 2003 (MEC, 2005), a decision problem is posed using a data table with four attributes and five alternatives. It then asks for the value of a linear function to be calculated which ranks the alternatives, so that a linear function can then be weighted to yield a specific objective. We broadened this activity, asking that the best car be chosen by applying two rules, the first taking into account the number of first places, and the second by deleting the high and low values and adding the remaining scores.

In this paper we analyzed the students’ results for the activity proposed. The study was done with secondary education students (abbreviated ESO in Spanish, Obligatory Secondary Education), most of them around 15 years of age.

The objectives guiding our research were:
. To see whether the students understood the decision rules and applied them correctly. The problem was easy to understand since the context, in which the problem was presented, to choose the best car, is familiar to most youngsters.
To see if the students could identify other situations or settings where the different rules of choice presented in the activity, or other applicable rules, can be used.

To observe whether the students used a mathematical process in other contexts requiring that a choice or decision be made, since the goal was to promote the development of the student’s ability to take a critical stance given the rules of choice that are applied in society.

To verify the validity of the activity designed via a questionnaire on fair decisions and well-founded judgments. This problem is part of wider research into the design and implementation of activities involving game theory and negotiation models using mathematics.

**Theoretical framework**

The salient points and the framework guiding our research were oriented around the studies conducted by the Organization for Economic Cooperation and Development (OECD) on the use of mathematics by students, which in the case of Spanish students shows their unwillingness to think beyond applicable procedures or routine problems in mathematics.

In Spain, one of the curriculum changes made to secondary education in 2007 involved the use of competencies in the curriculum. Mathematical competencies are understood as an individual’s ability to use mathematics to meet his needs as a constructive, committed and thinking citizen. This is expedited through the promotion of activities that involve the problem solving and discrete mathematics.

At the present time, problem solving is provided as part of the primary and secondary education curricula. A problem is defined as a conflictive situation that requires a solution for which an explicit procedure is not known beforehand. Many of the textbooks more commonly used by school children include guidelines for the solving of problems, almost always following the phases described by Polya (1945). These phases encourage the use of heuristics, considered as exploratory methods or algorithms used in the problem-solving process in which the solution is found by evaluating the progress made in the search for the final result. This is why heuristics are “golden rules,” conjectures, intuitive solutions or simply common sense. A problem is no more than a tool for thinking mathematically (Schoenfeld, 1992) and requires training individuals who can think for themselves and who can critique and reflect on the solutions.

The curricular content dedicated to problem solving, mandatory until the age of 16, offers the chance to work with discrete mathematics. This subject is, however, not well-known by practicing teachers, and the lack of available material and activities further serves to prevent this subject from being taught to the same level as in other countries (DeBellis and Rosenstein, 2004).

Discrete mathematics is one of the branches of mathematics which has seen the most progress in the 20th century as a result of computers. Different learning centers and associations, such as the Freudenthal Institute (Doorman et al., 2007), the National Council of Teachers of Mathematics (NCTM), and the American Mathematical Society (AMS) have promoted its incorporation into curricula to include experiments and research involving mainly the use of graphs, matrices and combinatorial, all of which are very useful in today’s mathematics (Kenney and Hirsch, 1991; Rosenstein et al., 1997). Material and ideas for its development have also been provided by the arrival of texts and projects such as COMAP (1988), based on a set of examples of the applications of mathematics most relevant to everyday life. The past decade has seen a proliferation of textbooks (Parks et al., 2000) and web pages along these same lines, such as http://www.dimacs.rutgers.edu.
Some Mathematics Education researchers regard discrete mathematics as a chance to revitalize mathematics in school (DeBellis and Rosenstein, 2004; Rosenstein et al., 1997). They see it as a chance for innovation and an opportunity to discover problems that are out of the ordinary (Goldin, 2004).

The problems associated with this field can also be used to build mathematical knowledge and to model situations, which help the student understand and control the world around him. Mathematical modeling can be viewed as a means for linking problem solving to the real world. While the term mathematical modeling itself may be in dispute, within mathematics it can be used to solve real problems and to include students in some of the phases involved with the modeling tasks (Stillman et al., 2007). The trend or perspective of models for teaching mathematics and solving problems is to teach real life situations in the classroom (Lesh and English, 2005).

It is within this framework and considering the points mentioned (development of mathematical and problem-solving skills and modeling of everyday problems) that we developed the decision-making activity involving several alternatives in a set context.

DESCRIPTION OF THE STUDY AND RESULTS

Design of the study

This paper details the results of a study conducted with 72 secondary school (ESO) students divided into four groups, with 35 third-year students in two groups, and 37 fourth-year students in the other two. Most of the students were between 15 and 16 years old. Also included are the results of an interview with a group of fourth-year students asked to justify their answers.

The activity used, “The best car,” was taken in part from PISA 2003 (MEC, 2005). A discrete decision problem was formulated by means of the data table with four attributes and five alternatives shown below. The students were asked to calculate the value of a linear function which ranks the alternatives so as to then weight a linear function and obtain a specified objective. The PISA activity was expanded by asking that the best car be chosen considering two rules (rule 1: number of top rankings and, rule 2: eliminate highs and lows and add).

<table>
<thead>
<tr>
<th>Car</th>
<th>Safety (S)</th>
<th>Fuel consumption (C)</th>
<th>Exterior design (D)</th>
<th>Interior space (H)</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ca</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>M2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>Sp</td>
<td>3</td>
<td>1</td>
<td>3</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>N1</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>Xk</td>
<td>3</td>
<td>2</td>
<td>3</td>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>

The scores are interpreted as follows:
3 points = Excellent  
2 points = Good  
1 point = Acceptable

For the student, the activity had three parts:

1. Question (a) consisted of evaluating a given simple linear function and choosing the best car:
   Total score = (3x S) + C + D + H
. Question (b) required creating a weighted linear function to achieve a specific objective:

\[ \text{Total score} = \ldots S + \ldots C + \ldots D + \ldots H, \]

which yielded alternative Ca as the best car.

. Question (c) required the students to apply two rules to select the best car according to each rule:

- Rule 1: number of first place rankings,
- Rule 2: sum of scores after eliminating highs and lows.

**Analysis of the information**

For the analysis of the results, the three questions in the activity were broken down into seven coded items as follows:

- T = Complete table,
- A = Value of a linear function,
- B = Assign weights to a linear function to achieve the objective,
- CR1 = “First place” rule,
- CR2 = “Eliminate highs and lows” rule,
- CE1 = “First place” choice,
- CE2 = “Eliminate highs/lows and add” choice.

The results of the data analysis are shown in four sections. The first includes the results on the success of each of the items. The second and third sections are dedicated to question (b) on the design of a function, and detail the results and strategies used by the students. The third section offers a comparison of the 72-student sample with the results of PISA for Spain and the OECD for questions (a) and (b). Along with the results there is a qualitative analysis of the interviews with one of the groups that took part in the study and whose students were asked to explain their solutions to the different questions in the activity, to express their opinions about the rules, to cite other situations where these rules are applicable and to name other rules of choice.

1. **Success of each of the seven items**

Table 1 shows the percentage of students who correctly solved each of the seven items comprising the three questions.

<table>
<thead>
<tr>
<th>Item</th>
<th>T (a)</th>
<th>A (a)</th>
<th>B (b)</th>
<th>CR1 (c)</th>
<th>CR2 (c)</th>
<th>CE1 (c)</th>
<th>CE2 (c)</th>
</tr>
</thead>
<tbody>
<tr>
<td>%</td>
<td>93</td>
<td>100</td>
<td>57</td>
<td>96</td>
<td>93</td>
<td>83</td>
<td>82</td>
</tr>
</tbody>
</table>

The results of the study with 72 students yielded a high percentage of correct answers. There were no noticeable differences between the groups of third- and fourth-year students. Ninety-three percent of the students completed the table they were given with the score for each car (item T). All the students managed to correctly solve question (a) and apply the rule, i.e. to find the value of a linear function (item A). Most, over 90%, applied rule 1 (CR1) and rule 2 (CR2). Over 80% also chose the correct car by applying the rules. The biggest difficulty was posed by question (c) on the weighting required to have a given car (Ca) be the best. Slightly over half of the students answered this question correctly.
The activity shown here and consisting of 7 items was part of a larger questionnaire with 23 items on game theory and decision making, and which was intended to analyze the development of critical thinking and reasoning skills in secondary school students. The validity of the activity was determined using the Rasch methodology (Linacer, 2007), which provides a correlation between students and items and which confirms that assigning or selecting weights so as to achieve a desired result is difficult when compared to the other items formulated in the questionnaire (www.ince.mec.es/pub/pismanualdatos.pdf pp. 64-81).

We now present a detailed analysis of this question.

2. Results of the question to weight a linear function
To analyze item B (assign weights to a linear function), we established a coding system for Blank, Wrong, Acceptable and Right answers, where “Acceptable” was used when the assigned weights yielded an equal score between the target car and another car. Table 2 shows the number of answers for each category.

Table 2: Results for questions (b)

<table>
<thead>
<tr>
<th>Categories</th>
<th>Blank</th>
<th>Wrong</th>
<th>Acceptable</th>
<th>Right</th>
</tr>
</thead>
<tbody>
<tr>
<td>N = 72</td>
<td>2</td>
<td>18</td>
<td>11</td>
<td>41</td>
</tr>
</tbody>
</table>

In the results in Table 2, note how the majority of students, 41, answered correctly. But one fourth of them, 18, gave a wrong answer and were unable to assign weights to a linear function so as to achieve the desired objective. If to these we add those students with blank or acceptable responses, a total of 44% of the students failed this question.

As concerns question (b), the reasons the students interviewed gave for choosing the weights are:
- They assigned a higher weight to the higher scores
- They assigned weights at random
- They used trial and error.

We believe the failure of these students, who are nearing the end of their obligatory schooling, to come up with a weighted linear function to yield a given objective could have negative repercussions in their future lives as members of society.

Next we analyze the correct and incorrect strategies used by students when assigning weights in this question.

3. Listing and analyses of strategies for linear weighting
Table 3 summarizes the strings of four weights provided by the students, and assigns them to the Right, Acceptable and Wrong categories, along with the value of the object function for each attribute or car and the number of students offering said choice. When a cell shows several weights, the number of students choosing those weights is shown instead of the values.

Table 3: Answer patterns according to strategies for achieving an objective

<table>
<thead>
<tr>
<th>Grade</th>
<th>Weight</th>
<th>Total Scores</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>Right</td>
<td>3 1 1 3</td>
<td>21, 16, 19, 18, 20</td>
<td>9</td>
</tr>
<tr>
<td></td>
<td>3 1 1 4</td>
<td>24, 18, 21, 21, 22</td>
<td>7</td>
</tr>
<tr>
<td></td>
<td>3 1 2 4</td>
<td>26, 20, 24, 24, 25</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>3 1 1 6</td>
<td>30, 22, 25, 27, 26</td>
<td>4</td>
</tr>
</tbody>
</table>
Of the 72 students, over half, 41, chose the proper weights which, as shown in Table 3, are needed to have the first value be the highest of the five. A total of 17 different weights were given as answers. The most frequent was (3 1 1 3), given 9 times, which results in a score of 21 for car Ca, and (3 1 1 4), appearing 7 times and resulting in a score of 24.

The most frequent strategy was to assign high values to S and H and low values to C and D. Note how some students changed the last 3 (weight of H) to a 4, possibly to obtain a larger difference between the two highest scoring cars. This same strategy seems to have led to other weightings such as (3 1 1 6) and (3 1 2 6).

The 11 cases graded as acceptable correspond to ties, and resulted from weights such as (3 1 2 3) or (2 1 2 3). Some students assigned weights to the attributes only to check the results with the alternatives they believed to be the most problematic, which may be why they failed to notice the tie.

The 18 students who answered incorrectly assigned weights like (3 2 3 3), (9 2 2 3) and other similarly nonsensical or absurd answers.

As already noted, it is troubling how so many students at or near the end of their mandatory schooling were unable to assign weights to achieve an objective. Differently stated, they were unable to analyze and work with data given to them so as to obtain the desired result.

4. Comparison with the PISA results

Table 4 shows the percentage of right answers for questions (a) and (b) given by the 72 students in the study, along with the results from Spain and the OECD. For question (a) we considered the result for item T, since in PISA the results of items T and A are combined.
Table 4: Comparison with the PISA results

<table>
<thead>
<tr>
<th>Question (a)</th>
<th>ESO N=72</th>
<th>Spain PISA</th>
<th>OECD PISA</th>
</tr>
</thead>
<tbody>
<tr>
<td>Question (a)</td>
<td>93%</td>
<td>71.4%</td>
<td>72.9%</td>
</tr>
<tr>
<td>Question (b)</td>
<td>57%</td>
<td>22.2%</td>
<td>25.4%</td>
</tr>
</tbody>
</table>

For the 72 ESO students, the results are noticeably better than those for the rest of Spain and also than those for other OECD countries. For question (b), however, on the assignment of weights or the prioritization of alternatives to achieve a specified outcome, all three gave unsatisfactory results. The task involves an important mathematical process and evidences a lack of conceptual and algorithmic learning in those countries where the test was administered.

During the interview, the students were asked to give their opinions about the rules and to name other situations where these selection rules are applicable. In their answers, several commented that the rules applied to question (c) did not seem fair since both resulted in the least safe car being given the highest score, although they do accept that, in the case of cars, it makes sense to pick the car with the highest score. Moreover, they were able to identify these two rules with real situations and to come up with contexts, such as sports competitions, specifically synchronized swimming, ice skating, diving, dance contests, surfing and some television game shows, where said rules apply. They had problems inventing rules however, and most simply proposed summing the scores.

It was also apparent from the interviews that this was a new problem for them. They did not, for example, associate the scenario with functions, i.e. they did not see the activity as involving mathematics, such as finding the value of a function, and so they were not aware of any transfer of knowledge (Santos, 1997). They approached the problem as a novelty that they had never encountered in mathematics class.

CONCLUSIONS AND REFLECTIONS
The results from the study with third- and fourth-year secondary school students showed how they all completed the table with the scores for each car. They carried out a routine procedure given a direct instruction. We found that they understood the decision rules and, as expressed by those interviewed, were willing to apply them to their lives, though they did not see a connection to mathematics. Almost half of the students in our sample experienced difficulties in finding the right weights, and also failed to use the proper notation. In brief, it may be said that the students were able to read a table and to find the value of a function, but faced serious difficulties when assigning weights to reach a stated objective and said they knew of no method that would guarantee success.

The conclusions drawn from the data for the countries participating in PISA are a cause for reflection, in that while the students know how to apply rules to a data table, they do not know how to manipulate the data to reach a specified goal. It may be deduced from the OECD data, which agree with our own, that certain aspects of the educational system as it now stands are failing. Students do not have a procedure or a linear strategy they can apply, as they do when calculating an average or an expected value. Other studies (Cobo and Batanero, 2004) have shown that a large number of students experience difficulties when calculating and interpreting weighted averages in statistics. The correct weight when calculating an average involves the ability to apply the distributive property when adding a set of numeric values. Another valid analogy to demonstrate this problem is the “knapsack problem” or the “subset sum problem,”
where given a weight or sum, the appropriate items have to be selected to fill the knapsack. This problem is NP-complete, but there are good heuristics that could be taught (Espinel, 1995). It seems students are not taught methods for selecting weights or values from among a given group to reach a total sum or weight.

We are currently designing and improving this and other activities so as to achieve the stated objectives of our research.

Of the three aspects considered in PISA (content, situation and competence), the “Best car” task is considered a public situation. It requires that the students resort to their mathematical understanding, knowledge and skills to evaluate the aspects of an external situation with repercussions in public life. Those competences or processes involved in the problem development are thinking, reasoning and argumentation (Niss, 2002). This is, in our opinion, a rich activity to take into the classroom (DeBellis and Rosenstein, 2004) since it covers a wide variety of mathematical fields, such as social choice, voting methods, social justice and the search for fair rules, and decision-making theory and the search for an optimum decision from a set of alternatives.

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Referentes:


A phenomenological study about mathematical problem solving is described. Eight pre-service mathematics teachers participated; six were studying to become teachers at elementary school 4th to 6th grades- and two at high school 7th to 12th grades-. The data was obtained through long interviews, thinking out loud problem solving sessions and retrospective interviews that took place immediately after the problem solving sessions. The objective of the long interview was to determine the participants' beliefs and declarative knowledge about this topic. The objective of the problem solving sessions was to determine the type of representation, strategies, and control processes that the participants use when solving problems. During the retrospective interview, the participants had the opportunity to reflect about their performance. These techniques allowed the investigators to obtain a comprehensive description of the phenomenon.

INTRODUCTION

Problem solving is recognized by some experts as a fundamental process for the mathematical development of students. Some of the most influential organizations in mathematics education have recognized the importance of problem solving in school mathematics (National Council of Teachers of Mathematics, 2000; American Association for the Advancement of Science, 1993). The Mathematics Program of the Department of Education of Puerto Rico, states in its mission that solving problems is important as an aim and a mean for the learning of mathematics. (DEPR, 2003).

Regardless of the recommendations and the importance recognized by experts and educational organizations, what happens in the mathematics classrooms is completely different. After finishing their education, some teachers ignore what they learned, and their practice reflects more their experience as students rather than what they were taught in their pedagogy classes (Skott, 2001, as cited by Liljedahl, et al. 2007). Some authors ascribe this situation to the teachers’ knowledge about the discipline (Ball, 1990; Leonard and Joergensen, 2002; Van Dooren, et al., 2003); others, to teachers’ affective and metacognitive factors, including their beliefs (Grows y Good, 2002; Liljedahl, et al. 2007; Mewborn y Cross, 2007).

Education schools prepare their students in three areas: discipline knowledge, knowledge of the fundamentals of education, and the teaching of the discipline. However, the time dedicated to the study and development of metacognitive strategies is minimal. We hope that this research can provide the insight required to make curricular changes, so that the study and development of the metacognitive processes can be included in the curriculum of prospective teachers.

REVIEW OF LITERATURE

Problem solving has been the object of many researches; however, there are still unanswered questions about it (Lester, 1994). The study of problem solving started with the publication of How to Solve It (Polya, 1945), which was followed by a series of research studies about the effectiveness of the use of general problem-solving strategies to learn mathematics. Later, research was based on the information processing theory with the objective of determining how the experts solve problems (Schoenfeld, 1985; Schoenfeld & Herrmann, 1982; Silver & Marshall, 1990). Recently, with the advent of the constructive learning theory, there has been a new wave of research that focuses in other aspects, such as the role of the metacognition, the students and teachers' beliefs, and affective and social influences when solving a problem (Garafalo and Lester, 1982; Hernández Rodríguez, 2002; Maqsud, 1997; Santos Trigo, 1995;
Schoenfeld, 1987, 1989, 1992; Swanson, 1990, 1992). It has been established that, even if the students possess the mathematical knowledge, it is very difficult for them to use that knowledge in new situations (Santos Trigo, 1995; Schoenfeld, 1985). Moreover, in few occasions they are able to use specific strategies to solve mathematical problems (Hernández Rodríguez, 2002). It also has been concluded that, in addition to the specific knowledge, students need other strategies to solve problems (Polya, 1945; Santos Trigo, 1995; Schoenfeld, 1985). More specifically, it has been established that students can benefit more if they learn general problem-solving strategies (Hembree, 1992; Lawson, 1990; Silver and Marshall, 1990).

Flavell (1976) defined metacognition as the knowledge that people have about their own cognition and the self-regulation processes of the cognitive processes. Later, this definition was expanded to include the students’ beliefs about themselves, about mathematics, about the task, and about the strategies required by a given situation (De Corte, Greer and Verschaffel, 1996; Garafalo and Lester, 1985; Greeno, Collins and Resnick, 1996; Lampert, 1990; Schoenfeld, 1987). Stemberg (1998) considered that metacognition is part of the human abilities and that it is indispensable in the formation of the discipline.

Lamper (1990) found that students believe knowing mathematics is about remembering and applying certain rules correctly in a problem, and that the only correct answer is the one the teacher gives. These beliefs have a generally negative effect in the way the students perform when solving mathematical problems. Schoenfeld (1987) stated that students’ mathematical beliefs are important to help or interfere in the process of problem solving. For example, he found that students thought that a mathematical problem has to be solved in less than ten minutes and this belief lead them to abandon it if they do not get a solution quickly.

Problem solving in pre-service mathematics teachers had been studied in different aspects. Bjuland (2004) did a study with 105 pre-service teachers in which they reflected on their own learning process when solving geometric problems in a collaborative way. Chapman (2005) conducted a qualitative study intended to determine the knowledge that pre-service teachers have about problem solving and the effects of the incorporation of reflection and inquiry processes to the improvement of this knowledge. Cadenas (2007) did a study intended to determine the lacks, difficulties and errors that pre-service teachers have in their mathematical knowledge previous the start of university studies.

Mathematics teachers’ beliefs and their relationships with the student learning process have been studied by Mewborn and Cross (2007). Teachers’ cognitive and metacognitive preferences have been studied by Leikin (2003), and Grouws and Good (2002). In regard to pre-service teachers, Liljedahl, Rolka and Rösken (2007) studied the affective aspects of problem solving, and Van Dooren, Verschaffel, and Onghena (2003) investigated the evolution of pre-service mathematics teacher’s cognitive preferences.

Lampert (1990) said that the school has a major responsibility in the development of student's beliefs about the meaning of knowing mathematics and how to do mathematics. These beliefs are created after so much time of looking, listening and practicing mathematics in schools. Since beliefs are mental constructions originated by previous experiences and social interactions, we can argue that student’s beliefs are mostly influenced by their teacher's beliefs. Part of the difficulty that students show with problem solving can be explained by their teacher’s beliefs about it (Goss, 2006; Mewborn and Cross, 2007). At the same time, the mathematics teachers’ beliefs are the result of their school experiences and the knowledge acquired as education students. The dialectic interaction between beliefs and professional evidence are the object of this investigation.
With regard to the representations that are used by pre-service teachers when solving problems, it has been found that the numeric-table representation dominates over the algebraic and geometric ones (Presmeg and Nenduradu, 2005). Moreover, Mousolide and Gagatsis (2004) found that teachers do have difficulties constructing geometric representations.

It is important to remark the importance of mathematic content in the problem solving process. Some authors claim that a good knowledge of the discipline is a great factor when dealing with problem solving (Ball, 1990; Cadenas, 2007; Leonard and Joergensen, 2002; Van Dooren, et al., 2003). Ma (1999) concluded that teachers that have a deep understanding about mathematical concepts can create a problematic situation related to the mathematical concept more easily, which implies that a deeper understanding of the mathematical concepts redounds in a better pedagogical knowledge.

To think about the beliefs and the way that pre-service teachers solve problems will allow the researchers propose educational environments that promote the construction of beliefs and knowledge that favor problem solving.

**RESEARCH QUESTIONS**

This investigation was guided by the following questions:

- What beliefs do pre-service mathematics teachers have about mathematical problem solving?
- Which kind of external representations (iconic or symbolic) do pre-service mathematics teachers use when solving a non typical problem?
- What kind of strategies (general or specific) do pre-service mathematics teachers use when solving a non typical problem and in which circumstances do they use them?
- How does self-regulation intervene during the different stages of problem solving?

**DEFINITION OF TERMS**

A non typical mathematical problem is a situation that has to be modeled to find an answer to a question that derives from the same situation and which solution is not straightforward (Parra, 1991).

Problem solving refers to the coordination of knowledge, previous experiences, and intuition in an effort to find a solution that is unknown (Parra, 1991). Operationally, it is the set of all written and verbal processes used by the student to find the answer to a problem.

The cognitive processes that will be studied in this investigation are the construction of the problem's representation and the strategy selected and used to solve a problem.

An external representation is a stimulus to the sense, generally in the form of drawings, diagrams, graphics, models or other formal symbolic systems (Janvier, Girandon and Morand, 1993).

A general strategy is a technique that can be applied to various knowledge domains and that serves as a guide to solve a problem. Some general strategies are trial and error, finding a pattern, constructing a table, using analogies, using auxiliary elements and backward working. A specific strategy is a technique that can be used to solve a problem in a specific domain.

Metacognitive processes include the beliefs and the processes of self-regulation and control. A belief is a made-up explanation that a person has about a specific field of knowledge and that determines the way the person conceptualizes and fulfills on it (Schoenfeld, 1992). A belief can be about itself (De Corte, Greer and Verschaffel, 1996), the area of study - mathematics, in this case- (Greeno, Collins and Resnick, 1996) or the task to be done (Garofalo and Lester, 1985).
The **self-regulation or control** is an ordered process used by a person to control its own cognitive activity, and, in this way, ensure the accomplishment of the cognitive objective (Schraw and Graham, 1997). A person that can control its cognitive activity can make predictions, elaborate a plan before starting to solve a problem, pay attention to all the components of a problem, question the process, value the products and the efficiency of the execution, and review, change, and abandon unproductive strategies or plans (Garofalo and Lester, 1985; Schraw and Graham, 1997).

**METHODOLOGY**

In the present study, a constructivism point of view is assumed. This considers that human beings build their own knowledge, and that cognitive and metacognitive processes take part in the construction of such knowledge (Flavell, 1976; Noddings, 1990; von Glasersfeld, 1990). In addition, it is also recognized that some social and emotional factors intervene in the construction of knowledge (Greeno, Collins, and Resnick, 1996). To have access to the cognitive and metacognitive processes, different methodologies were used to describe what was happening in the minds of the participants. The research design used responds to the necessity of gaining access to the field of the perceptions to explore what is happening when non typical problems are solved from the participant's perspective.

This is a phenomenological study about cognitive and metacognitive processes that pre-service mathematics teachers exhibit about solving non typical mathematical problems. It was intended to find deep meanings, understandings and attributes of the phenomenological target that was studied. The meaning that various people ascribe to the concept or phenomenon is described (Creswell, 1998; Moustakas, 1994), the experiences that people have had with the phenomenon are explored, and the essential structure or invariant in which underlies the meaning of the experience is outlined. In this way, the intentionality of the conscience is described where the experiences contain the external appearance as well as the internal conscience (Moustakas, 1994). From this point of view, the phenomenon gains meaning through people experiences with it. To have access to the essence, product of the range of interactions that the people have had with the phenomenon, we analyzed the memory, the image and the significance that they attribute to it (Creswell, 1998; Morse, 1994).

**Participants**

The participants were university students enrolled in the teachers formation program of a public university in Puerto Rico, specifically, those who majored in mathematics education at elementary level or secondary level. Participation was on a voluntary basis. A public announcing was done to the university community, requesting the participation of volunteers; eight people applied, all of them females. The candidates assisted to an orientation, in which the nature of their participation was discussed, as well as the observance, by the investigators, of the university rules concerning the protection of human rights when students participate in an investigation.

**Data collection**

The procedures used to collect the information were descriptive and qualitative, designed to describe a wide range of internal and external activities. The techniques used were: long interview, thinking-out-loud problem solving session, and a retrospective interview right after the problem solving session. These techniques let the participants reflect about the theme, which will contribute to their formation as teachers.

The long interview allowed the access to the meanings that participants had about
problem solving and it was possible to describe the beliefs that they have about this process. After that, the participants solved four non typical mathematical problems. The participants were instructed to solve the problems thinking out loud. This technique allowed the access to the cognitive and metacognitive processes the participants went through. Immediately after each problem solving session, a retrospective interview was made in which the participant had the opportunity to reflect on their execution in the problem solving process. Thus, not only was explored what was happening in the participant's minds, but also the problem solving execution and the participant's thoughts about it. In this way, there were three sources of information that allowed the triangulation of the data, allowing the researches to conclude which kind of representation, strategy, metacognitive process of control and beliefs the participants have about solving non typical mathematical problems.

Problems

The problems used have the characteristic that they are sufficiently challenging to produce metacognitive behavior, but at the same time could be solved by the students with the mathematical knowledge they had from their mathematical classes (Goos and Galgraith, 1996). In addition, the problems can be represented in different ways and solved using diverse strategies. The problems that were used for this analysis were problem 2 and 4.

PROBLEM 2

A square and a rectangle have the same area. The square diagonal has a longitude of $8\sqrt{2}$ inches. If the width of the rectangle is 4 inches, what is the length of the rectangle?

PROBLEM 4

A candy sale is organized with the purpose of raising funds for the Children Cancer Association. Olga, who is engaged in the cause, wants to sell 27 chocolate bags. There are two different kinds of chocolate: with almond and with strawberry. Each bag of chocolate with almond has 8 bars and each bag of chocolate with strawberry has 9 bars. If Olga has a total of 232 chocolate bars, how many bags of each kind of chocolate does Olga have?

Analysis

The analysis of the collected data was enriched by the comprehension reached by the investigators, and complemented by the review of literature and their experience as professors and investigators. All the interviews were made and transcribed by the investigators. The fidelity of each transcription was corroborated with the audio records and simultaneous reading of what was transcribed.

Particularly, the analysis that the investigators did of the long interviews let them find an answer to the students’ beliefs question. The retrospective interview analysis let them determine which self-regulation processes were used when the participants solved each problem and helped them complement and contrast all the information obtained in the long interviews and problem solving sessions.

On the other hand, the analysis of the problem solving session provides plenty information about how the students construct the representation of each problem, the strategies that they use and the self-regulation strategies they showed in the solution process. The investigators used the audio record of each problem solving session, all the computations the participants made during the process and the transcriptions of their observations as data to analyze.

RESULTS AND DISCUSSION

The results obtained are organized following the research questions. The extensive interviews were used to establish the students’ beliefs about mathematical problem solving, and
the analysis of the solutions of problems 2 and 4 to establish the representations, the strategies and the processes of self-regulation used by the participants. The retrospective interviews were used to triangulate the information.

Beliefs

In general terms, the participants consider themselves very good mathematical problem solvers. This indicates great self-confidence although in many occasions they expressed having difficulties solving the problems. They attribute their good disposition mainly to affective reasons. The majority declared that problem solving represents a challenge and it motivates them; other reasons for their good disposition were that they were interested in mathematics or that they liked to solve problems.

The participants characterized a mathematical problem as an uncertain situation because they do not know the subject matter or the method needed to solve it. A problem requires deeper analysis; several pieces of knowledge take part and must be used simultaneously in the process of solving a problem. In contrast, they already know what to do when it comes to solving an exercise, since exercises are solved with some well-known algorithm.

Just as in the investigation of Chapman (2005), the majority point out that the steps used to solve a problem are: reading the statement, identifying the given data, determining what is asked and solving it. The participants assigned greater importance to the understanding of the problem and less to the process of solving it. Still more, some participants assigned great importance to the reading of the problem, because they think that the way to solve the problem is ciphered in the statement.

The students declared preferring the arithmetical and algebraic strategies over the graphical strategies. All the participants indicated that they verified the problem to know if the answer were correct, however this was not observed when they solved the proposed problems.

Some indicated that, when they are solving a problem and do not know how to follow, they returned to read and “reread” the statement. Others indicated they abandoned it temporarily and returned to it later on. One student indicated that she reviewed the notes to see if she had solved a similar problem previously, another one mentioned that she tried to get help. Half of the students indicated that “they analyzed” the problem when they did not know how to follow ahead. For them, a problem is difficult when: they cannot solve it in the first attempt, or they do not understand it when they read it the first time. It is also difficult when it contains too much information or when different operations are needed in the solution process.

When asking how they considered that problem solving should be taught at school, the majority indicated that it must be integrated more frequently in class. They also point out that problem solving should not be a separate or isolated topic; instead it must be related to daily life events. Also they indicated that the teachers must give the students more problems to solve, that is, more frequent practice, thing that many of them did not get when they were students. This aspect agrees with the obtained by Grouws and Good (2002), which found that problem solving was not very frequent subject in the classes of mathematics of the observed teachers.

One of the participants narrated a negative experience that she had had with a university professor when she was solving a problem that involved roots. Nowadays it is very difficult for her to solve problems that includes these, even more, she feels uneasy whenever she faces a problem with that include roots. In fact, she could not solve problem 2, which included a squared root.

Representations

Immediately after reading problem number 2, the students used graphs to represent it.
They drew a square, a rectangle and some marked the measurement of the diagonal of the square. In addition, three students represented in iconic form the condition that the area of the square was equal to the area of the rectangle. Nevertheless, only two could establish the connection between the graphical representation and the algebraic one, which limited the use of this one. This last point converges with the indicated by Gagatsis, Elia, and Kyriakides, 2003; as cited by Mousoulides and Gagatsis (2004), in the sense that the pre-service teacher that participated in their study could not make the connection between the graphical and the algebraic representation.

Several difficulties in the representation of the problem appeared. These can be classified as difficulties that came from the invention of conditions from the data, the omission of data that the problem provided, and others whose origin is mathematical. First of all, three students used \(8\sqrt{2}\) as the diagonal of the rectangle. Second, a student did not take into account the piece of information regarding the fact that the two figures had the same area; this prevented her from solving the problem. Finally, the main mathematical error in the representation was that the students thought that the measurement of the diagonal of the unit square is one. This made them construct a square whose sides measured the same as the diagonal.

The representation of problem four occurred in different form. Half of the students underscored the relevant information in the statement of the problem; the other half rewrote it in the worksheet. The majority used numerical representations, only 2 made algebraic representations. After reading the problem, the students began to conduct arithmetical operations with the numbers that the statement provided, which indicates that there is no understanding of the problem immediately after the reading.

The representation used is privileged by the situation of the problem. In problem number two the graph prevailed, whereas in problem number four the numerical one, nevertheless, the participants could not connect this first representation to the mathematical content that allowed them to solve the problem. Once the initial representation was constructed, the participants did not change it, which shows little flexibility to change plans, although in several occasions the students returned to reread the statement.

**Strategies**

To solve the second problem, three participants tried to construct the figures from the diagonal. None were successful because of the errors they had committed in the initial representation of the problem. Two participants used the Pythagorean Theorem, one succeeded and the other not, since she used it in a rectangle, in which she had labeled \(8\sqrt{2}\) as the measure of the diagonal. Two students did not make any attempt solve the problem, nevertheless, one guessed an answer. The other student indicated the process that she would have used to solve the problem; nevertheless, she indicated that she did not know how to calculate the side of the square.

With respect to the fourth problem, seven of the participants used the strategy of trial and error. From these, three arrived at the correct answer. They used “educated estimates”, that is, they tried some set of possible values and adjusted them according to the results they were obtaining, eventually arriving to the correct answer.

Five participants performed calculations with the numbers given in the problem. They carried out operations such as \(232 \div 2 = 116; 116 \div 8 = 14.5; 116 \div 9 = 12.88\) or \(27 \div 2 = 13.5\), with the hope that the results would fit some of the given information. This agrees with the findings of Kieran, 1992; and Linchevski and Herscovics, 1996; as cited by Van Dooren, Verschaffel, and Onghena (2003), whom indicated that the students preferred to perform
arithmetical operations with the known numbers, the meaning of such operations remaining invariably connected to what the students perceive to be the context of the original problem.

One participant used the specific strategy of system of equations; in particular, she set up a system of 2 linear equations with 2 variables, obtaining the correct answer. This participant was majoring in secondary education in mathematics. This finding agrees with the obtained one Van Dooren, et al. (2003), whom indicated that secondary school pre-service teachers prefer to use of algebra.

Several investigators have documented the difficulties that the students have when they face algebra for the first time and, in specific, the solution of algebraic problems (Filloy & Sutherland, 1996; Herscovics & Linchevski, 1994; Kieran, 1992; Lee & Wheeler, 1989; Sfard & Linchevski, 1994; as cited by Van Dooren, et. al, 2003). Some educators and investigators have suggested that a way to solve these difficulties is "to algebrazied" the elementary mathematics curriculum (Ainley, 1999; Davis, 1985; Discussion Document for the Twelfth ICMI Study, 2000; Kaput, 1995; Swafford & Langrall, 2000; Vergnaud, 1988; as cited by the authors). They argued that, early in the school mathematics education, the arithmetical activities can and must gradually be attended with an algebraic meaning with the purpose of emphasizing the inherent algebraic characteristics. Incorporating this recommendation to the curriculum of pre-service elementary teachers will possibly help them extend their repertory of strategies to solve problems successfully.

**Self-regulation**

During the problem solving sessions the participants showed little metacognitive activity, for example, they did not express their familiarity with the problem, nor stated its level of difficulty. In the case of the geometry problem, a student declared with disappointment “this is a geometry problem” when reading the problem. In the retrospective interview it was possible to verify that she had trouble with the subject. Another student stated that everything related to radicals caused her anxiety due to an unpleasant experience she had when she was learning them. The other students did not indicate their confidence (or lack of it) of solving the problem.

In few occasions the participants showed evaluation strategies of the representation or the strategy they were using. For example, very few participants changed the initial representation and even fewer changed the strategy that they selected initially even if it did not produce the awaited results. The evaluation of the progress towards the solution occurred more in problem four than in problem two, since the students used the values of the statement of the problem or some that they considered important to the solution. Another element that was used to verify the progress towards the solution of the problem was the appearance of decimal numbers in the results of the operations. In this aspect, it is important to stress that although the initial performed operations did not have sense, these helped them start making use of the trial and error strategy with initial values closer to the actual solution.

Having taken a geometry course did not help them solve problem number two. Similarly, the familiarity with problem number four did not help them arrive to the correct answer. This is explained since it is difficult for students to apply their mathematical knowledge to a novel situation (Hernandez Rodriguez, 2002; Santos Trigo, 1995; Schoenfeld, 1985; Selden, Selden, Mason, 1994).

It was observed that some of the participants who had been more time working in the problem tend to ignore the initial conditions. This can be explained in two ways: they are approaching the time limit that they are supposed to expend on it or for them it is indispensable to give an answer (Schoenfeld, 1989).
CONCLUSIONS

1. The participants characterized the mathematical problem as an uncertain situation because they do not know what it is about or the method in question to solve it. A problem requires deeper analysis; several pieces of knowledge take part and must be used simultaneously in the process of solving a problem. In contrast, they already know what to do when it comes to solving an exercise, since exercises are solved with some well-known algorithm.

2. The participants declared to prefer the arithmetical and algebraic strategies over the graphical strategies.

3. All the participants indicated that they verified the problem to know if the answer was correct, although this was not observed in the problem solving sessions.

4. Most of the participants indicated that problem solving must be integrated more frequently in mathematics classes. Problems solving should not be studied as a separate or isolated topic, and it must be related to daily life situations.

5. Participants assigned great importance to the reading of the problem because they think that the way to solve it is ciphered in the statement.

6. Participants used a graphical representation for problem number two and a numeric representation for problem number four.

7. There were several difficulties in the representation of the problems. These can be classified as difficulties that came from the invention of conditions from the data, the omission of data that the problem provided, and others whose origin is mathematical.

8. With regard to the geometry problem, three participants tried to construct the figures from the diagonal. This strategy was not successful because of the errors they committed in the initial representation of the problem.

9. Two participants used the Pythagorean Theorem to solve problem number two. One was successful and the other not, since she used it in the rectangle and she had labeled its diagonal as $8\sqrt{2}$.

10. With regard to problem number four, seven of the participants used the strategy of trial and error. From these, 3 arrived at the correct answer. The method used was “educated rough estimates”, that is, they tried some set of possible values and adjusted them according to the results they were obtaining, eventually arriving to the correct answer.

11. The participants showed little metacognitive activity. They did not express their familiarity with the problem, nor the level of difficulty of the same. They did not indicate their confidence to solve the problem.

12. In few occasions, the participants showed evaluation strategies.

13. It is difficult for the participants to apply the mathematical knowledge to solve unfamiliar situations.

14. Participants possessed declarative knowledge about problem solving; however, it was difficult for them to use it to solve the problems posed.

Educational implications

Pre-service mathematics teachers must be exposed frequently to problem solving in their mathematics classes, so that they develop the necessary skills to solve and teach appropriately to their students. The use of diverse representations should be stimulated to fortify the connection between them, so that they can use the one that is suitable at the appropriate time.

Finally, it is essential to foment the use of algebraic strategies in the pre-service
elementary mathematics teacher, so that the arithmetical activities can be attended with an algebraic meaning.

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Teachers’ beliefs about mathematical problem solving, their problem solving competence and the impact on instruction: A case study of three Cypriot primary teachers

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Abstract
In this paper we report a case study of three Cypriot primary teachers, with respect to their mathematical problem solving beliefs and competence, and the impact of these on instruction. Semi-structured interviews were carried out with the teachers. Each of them was invited to solve a purely mathematical non-routine problem and explain simultaneously the solution process. The teachers prepared a lesson based on the problem and taught it to their classrooms. Our findings suggest that teachers’ mathematical problem solving beliefs, competence and instructional practices are in a complicated relation that cannot be explained in terms of cause-and-effect.

Introduction
Mathematical problems and problem solving
In starting this literature review it is important, particularly as the word problem, even within the domain of mathematics education, frequently means different things to different people (Borasi, 1986; Blum & Niss, 1991; Nesher, Hershkowitz & Novotne, 2003; Wilson, Fernandez & Hadaway, 1993; Goos, Galbraith & Renshaw, 2000), to consider how it is defined. A common definition is that a mathematical problem presents an objective or goal with no immediate or obvious solution or solution process (Blum and Niss, 1991; Schrock, 2000, Polya, 1981; Nunokawa, 2005). In summarising the work of Schrock (2000) and Wilson et al (1993) we suggest that a mathematical problem must meet at least three criteria; individuals must accept an engagement with the problem, they must encounter a block and see no immediate solution process, and they must actively explore a variety of approaches to the problem.

According to Chapman (1997) problem solving means different things to different people, having been viewed as a goal, process, basic skill, mode of inquiry, mathematical thinking and teaching approach. However, most research in the area seems to regard problem solving as the process of achieving a solution (Chapman, 1997; Blum & Niss, 1991; Boekaerts, Seegers & Vermeer, 1995; Franke & Carey, 1997; Hart, 1993). Famously, Polya (1981.ix) described it as a means of “finding a way out of difficulty, a way around an obstacle, attaining an aim which was not immediately attainable” and it is on this conception that we focus our work.

Various writers, including Polya (1945), have developed frameworks for analysing the problem solving process. Polya’s model comprises the four phases of understanding the problem, devising a plan, carrying out the plan, looking back. Other models, frequently based on Polya’s, include Kapa’s (2001) six phase and Mason et al’s (1985) three phase. The latter suggest that problem solving comprises entry, attack and review. However, space prevents a lengthy discussion on the details of these models and their similarities and differences, although it is our view that their resonance with Polya’s is close and not difficult to discern. From the perspective of this study, we tend towards Mason et al’s (1985) model as their
attack phase appears not to necessitate a predetermined plan in the manner of Polya’s devising and carrying out a plan.

In the context of Cyprus, the location of this study, much research on problem solving has been undertaken over the last few years (Christou & Philippou, 1998; Elia & Philippou, 2004; Gagatsis & Elia, 2004; Gagatsis & Shiakalli, 2004; Nicolaidou & Philippou, 2003). However, the focus of every study has been on students’ understanding, beliefs, abilities and attainment. Such work, when seen against the international trend for teacher-focused research on problem solving to address instructional effectiveness rather than teacher competence (Chapman, 1997, Thompson, 1985) or the impact of beliefs (Thompson, 1984) highlights well a field ripe for development.

In this paper we report on the mathematical problem solving beliefs and competence, and the impact of these on their instructional choices, of three Cypriot primary teachers. Beliefs, and their impact on teachers’ instructional choices, have been the subject of extensive investigation in mathematics education. However, despite this extensive research, ambiguity regarding terminology has caused confusion to the area (McLeod, 1988; Pehkonen & Pietila, 2003; Törner, 2002). In this paper we draw on Raymond’s (1997: 552) definition in which teachers’ mathematics beliefs, in relation to the nature of mathematics, its teaching and learning, refer to their “personal judgments about mathematics formulated from experiences in mathematics”.

**Methodology**

A case study investigation was undertaken in March, 2007. The participants, who were all teaching 11 and 12 year-old children at the time, were given the pseudonyms Mrs Antigoni (22 years of teaching experience), Ms Electra (newly appointed teacher) and Mr Orestis (second year of teaching). Initial semi-structured interviews focused on four thematic areas relating to how colleagues viewed themselves as teachers of mathematics, their espoused beliefs about the nature of problem solving, their perceptions of themselves as problem solvers and, finally, their beliefs about the management of problem solving in classrooms. After the interviews, each teacher was invited to solve a mathematical problem and explain simultaneously the solution process. The problem presented was the following:

“On the grid paper you have been given, each little square is equal to one square unit. How many isosceles triangles can you make which will satisfy all of the following three criteria?

1. The area must be nine square units.
2. One of the vertices is at the given point.
3. The other two vertices are on grid points too.”

Problems of this nature, which are non-routine and purely mathematical (Blum & Niss, 1991) are never found in the National Textbooks of Cyprus.

The final phase involved each teacher in preparing and delivering a lesson based on the same problem. The lessons were observed with the ways in which the teachers presented the problem, managed the classroom during problem solving and their approach to responding to students’ questions being the primary foci.

The aim was to ‘sketch portraits’ of the three participants, regarding their beliefs on problem solving, their problem solving competence and the impact of the former on their instructional practice. The work of Alba Gonzalez Thompson (Thompson, 1984) was particularly
influential. Data analysis was framed by the three-phase problem solving model of Mason et al. (1985). The framework used for analysing classroom observation data (teachers’ instructional practices during the teaching of the problem solving activity) was chosen *a posteriori* since we did not want to apply predetermined categories. For this purpose, six out of the ten mathematical didactics (table below), proposed by Andrews (2007), were used. The mathematical didactics are teaching strategies used by teachers in order to facilitate students’ understanding.

<table>
<thead>
<tr>
<th>Activating prior knowledge</th>
<th>The teacher explicitly focuses learners’ attention on mathematical content covered earlier in their careers in the form of a period of revision as preparation for activities to follow.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Explaining</td>
<td>The teacher explicitly explains an idea or solution. This may include demonstration, explicit telling or the pedagogical modelling of higher level thinking. In such instances the teacher is the informer with little or no student input.</td>
</tr>
<tr>
<td>Sharing</td>
<td>The teacher explicitly engages learners in the process of public sharing of ideas, solutions or answers. This may include whole-class discussion in which the teacher’s role is one of manager rather than explicit informer.</td>
</tr>
<tr>
<td>Coaching</td>
<td>The teacher explicitly offers hints, prompts or feedback to facilitate their understanding of or abilities to undertake tasks or to correct errors or misunderstandings.</td>
</tr>
<tr>
<td>Assessing or evaluating</td>
<td>The teacher explicitly assesses or evaluates learners’ responses to determine the overall attainment of the class.</td>
</tr>
<tr>
<td>Questioning</td>
<td>The teacher explicitly uses a sequence of questions, perhaps Socratic, which lead pupils to build up new mathematical ideas or clarify or refine existing ones.</td>
</tr>
</tbody>
</table>

Table 1: Mathematical Didactics

**Sketching the three portraits**

**A. Mrs Antigoni**

*Beliefs about teaching/learning mathematics and about herself as a mathematics teacher*

For Mrs Antigoni, mathematics is the most important lesson in the primary curriculum. A ‘good’ mathematics teacher should give students many examples so that “they can adopt some procedures easily in order to solve similar problems”. Also, she should accept critique, either good or bad, in order to improve. She believes that all of her mathematics teaching aspects require improvement. However, she does not make any allusions to how she could improve her mathematics teaching. She only refers to external factors that could change and therefore improve the conditions under which she teaches (i.e. wishes that children would concentrate more during the lessons and that, pupils would have greater self-confidence). She claims that the quality of her teaching would be improved if the number of pupils per classroom decreases.

*Beliefs about the nature of problem solving*

A mathematical problem is “a procedure, which requires you to discover which information is given, to rank the given points, to find what the problem asks you to do and then solve it”. The given information should be clear and accurate so that all the children understand what has to be done. Problem solving is the achievement of a goal, either set by the problem solver or by others. Teaching via problem solving can be feasible only if children know the mathematical procedures. Otherwise, if students do not know how to solve procedural exercises, they will be very disappointed and they will not have the motivation to try and
solve more difficult problems. Both finding a problem’s solution and the journey towards the solution are important. The correct answer matters a lot since “in the future, students will be asked to take mathematics exams, for which they have to achieve a high mark. If the steps they follow are correct but still the answer is wrong, then they will not attain a good grade”.

**Beliefs about her competence as a problem solver**
Mrs Antigoni views herself as an average problem solver and is bored of problems which require a more advanced level of thinking because she has been tackling simple problems with simple solutions throughout the years of her teaching experience. Sometimes she views problems in a superficial way and often has to revisit a problem many times in order to understand what is required for its solution. Practice is for her the only way of improving her problem solving abilities. Difficult mathematical problems are a cause of anxiety to her. When she has to solve an advanced problem she tries to relax and concentrate on the given information. When she faces difficulties during the solution process, she either recalls previous knowledge regarding the solution of similar problems or tries to solve the problem algebraically. If the problem is to be taught to primary school children, she tries to simplify the solution, because “children in primary schools do not know algebraic equations”.

**The management of problem solving in her classroom**
The school programme, according to her, does not allow any time for problem solving activities in classroom. Students should not spend more than three to five minutes on a problem. Her mathematics teaching is based on the national textbooks of the Ministry of Education and Culture. She brings mathematical problems in her classroom she creates herself. It is easier and more practical for her to write problems similar to those in the national textbooks rather than to search for them on the internet or in books. Most of her students feel nervous when they have to solve mathematical problems. Only few of them will try to reach a solution while the rest quickly abandon every effort. When students face difficulties during problem solving, she tries to give them some hints that will help them to proceed. Students should be encouraged by the teachers with expressions like “well done, you are on the right track. Now try to think what else should be done”.

**Solving the problem**
Mrs Antigoni’s *Entry* to the problem began by acknowledging the central goal of the activity (finding triangles that satisfy all three criteria) and by introducing the formula for the area of triangles. After finding four of the 12 triangles of the group of “base 6, height 3”, Mrs Antigoni began examining another case instead of finding the rest of the triangles in the same group. Soon after that, she demonstrated her boredom and was ready to abandon the problem. She continued her effort, mostly because of my interventions. Eventually she found all 36 solutions, though with great difficulty. A wide range of emotions were observed; from her initial enthusiasm to the desire to abandon the problem.

**The lesson based on the problem**
Mrs Antigoni’s lesson was a procedural explanation of the problem by the teacher. Children were given very little opportunities to demonstrate their thinking since the teacher was explicitly explaining every step during the teaching period. Two mathematical didactics were observed, *Activating Prior Knowledge* and *Explaining*.
Activating Prior Knowledge
At some point in the middle of the lesson, she asked students to bring to mind a previous lesson they had on symmetry. She suggested that students could use the rules of symmetry to find the triangles.

Explaining
Generally speaking, the lesson was based on the teacher’s explanations and step-by-step guidance. Selectively, I present the two following selected incidences which confirm and justify my claim:
(a) When children were given the problem, the teacher told them to start by using the formula for the area.
(b) A student suggested that there could be a triangle with base 4 units and height 4.5 units. The teacher told him that they were to work only with integers and that decimal numbers could not be used. She did not offer any further explanations on this.

B. Ms Electra
Beliefs about teaching/learning mathematics and about herself as a mathematics teacher
Mathematics is for Ms Electra a very important lesson, since it is in the primary and secondary curricula of every educational system around the world. Its importance can be seen in daily life due to its many applications, like money, measurements and so on. She feels able to teach mathematics only if she is well prepared. She claims that she does not actually know mathematics, since she lacks advanced knowledge in this field. A ‘bad’ mathematics teacher is one who teaches procedurally and does not allow children to build their own conceptual understanding. A ‘good’ mathematics teacher approaches mathematics teaching in an interdisciplinary way, and tries to facilitate students’ learning and to make children appreciate mathematics. She wishes she would feel more secure in teaching lessons based on non-routine problems.

Beliefs about the nature of problem solving
A mathematical problem is like every other problem; it is a situation that requires a solution. This solution can either be approached through a simple procedure or certain strategies have to be employed. “There is a wide range of mathematical problems; at one end of the scale, there are very simple ones and at the other end, there are some which are much more complicated”. A mathematical problem has a verbal and a numerical part. The ways the given information and the question of the problem are expressed comprise its verbal part. The numerical part is included in the verbal, in the form of numbers and symbols. Problem solving is the procedure during which the problem solver understands the problem, separates the given information from what he/she is asked to achieve and uses all the elements of the problem in order to reach a solution. Teaching via problem solving is a method of direct approach of a mathematical concept and is a way of connecting a mathematical concept with daily life situations. Teaching via problem solving can be used as a way for introducing new concepts. The journey towards the solution of a problem is much more important than the solution itself. The solver has to find her/his own way which will eventually lead her/him to the solution. Only then has the problem solver really understood the problem and its solution and can explain her/his journey to others.

Beliefs about her competence as a problem solver
Ms Electra labels herself as an insecure problem solver. As she claims, “solving problems is not something I enjoy doing and furthermore I do not have the skills for it”. If she does not have to solve a difficult mathematical problem then she is not interested in solving it. She
would only try to solve a difficult problem if she had to present it to students and that would cause her anxiety. Mathematical problems in particular make her nervous, but "if I really had to solve it, I would try and use every possible means, because I am a really stubborn and proud person". When she faces difficulties during the process of solving a problem, she either asks for advice from someone more experienced or she temporarily abandons the problem until she relaxes and then revisits it and keeps trying until she reaches a solution.

The management of problem solving in her the classroom
Time parameters set by the school programme do not allow much time to spend on problem solving activities, states Ms Electra. Most of the problem solving activities she uses concern the introduction of new mathematical concepts. She does not spend time on problems if they are not related to a particular mathematical concept she wants to teach. The time a student should spend on a problem depends on her/his abilities and on the degree of difficulty of the problem. The majority of the students in her classroom do not like solving problems due to the insecurity children feel. Her class do not respond with enthusiasm when she presents them with mathematical problems. When her students face difficulties during the process of solving a mathematical problem, she helps each student independently. She asks her students questions that will help them to think rationally. She does not directly explain to children what to do; she prefers that they discover the way by themselves. Suggesting her students work in pairs is another method she uses. Students can learn a lot through discussion and exchange of ideas while working with mathematical problems.

Solving the problem
Ms Electra seemed a little nervous when she was told she would be given a mathematical problem. Nevertheless, during the process of solving the problem, she managed to control her initial anxiety and successfully navigated through the problem’s difficulties. As soon as she realised that the initial way she was following was not appropriate, because of the three criteria, she revised her thinking and applied a new way. After discovering a pattern, she easily generalised her ideas and found all 36 solutions.

The lesson based on the problem
Ms Electra’s lesson was mostly based on students’ sharing and explaining their ideas to their classmates. Also, Ms Electra was particularly interested in offering individual and group feedback and assessing her students’ understanding. This was achieved either through a group discussion, where the teacher was managing rather than explaining, or through questioning, where the teacher used to challenge her students with questions. Analytically, selected incidents confirming the mathematical didactics employed by Ms Electra are presented below.

Sharing
(a) A student argued that the height that begins from the vertex between the two equal sides of an isosceles triangle divides the triangle into 2 equal parts. The teacher asked the student to demonstrate his ideas on the board and explain it to his classmates.
(b) At a point where most of the students had found all of the triangles with base 2 and height 9 and base 6 and height 3, the teacher asked if there was anyone who would like to explain analytically her or his way of thinking to the others. One student went to the board (where a grid paper was presented though an overhead projector) and explained to her classmates how she found all the triangles with base 6 and height 3. In the meantime, the other students asked that student questions on some of her steps.
Questioning
The teacher asked the students to number the characteristics of isosceles triangles.

   Student: The sum of their angles is 180°.
   Ms Electra: Is this a characteristic that only isosceles triangles have?
   Student: No, it applies to all triangles.
   Ms Electra: How important do you think this information is for our problem?
   Student: We don’t need it.

Assessing and coaching
When students were working either individually or in groups, Ms Electra went round the class and asked them to explain what they were doing. Her help and feedback was very encouraging. When the students faced difficulties, she challenged them with hints that would lead them to discover their own way.

C. Mr Orestis
Beliefs about teaching/learning mathematics and about himself as a mathematics teacher
Mr Orestis views mathematics as a way of thinking. Mathematics learning and the acquisition of particular techniques/information are for him totally different. The learning of mathematics can help students in their daily lives. More importantly, though, mathematics helps the learner in developing a more organised way of thinking. He feels confident in teaching primary mathematics but he would like to improve his mathematics teaching abilities as regards the introduction of new concepts. He wishes he knew better ways of developing the conceptual understanding of his students, because “a good mathematics teacher is one who helps students in building a conceptual understanding of mathematics rather than finding answers mechanistically”.

Beliefs about the nature of problem solving
A mathematical problem is a situation which leads the solver “on a mental adventure”. The solver has to use the given information methodically in order to arrive at a certain solution. There must not be a standard way to the solution and problem should be clearly posed so that it does not lead the solver to misunderstandings. Problem solving is the successful manipulation of the given information in order to find a solution. Teaching via problem solving is not feasible for all mathematical topics, despite the fact that it should be at the centre of teaching and learning mathematics. The journey towards the solution of a problem is much more important than its solution. It does not actually matter if the solver did not reach a correct answer because of some procedural mistakes when using the numbers and so on. “The emphasis is on the mental processes required in order to arrive at the solution”.

Beliefs about his competence as a problem solver
Mr Orestis feels confident at solving mathematical problems, and particularly those which refer to primary mathematics. Experience is for him a very important factor for someone to improve her/his problem solving abilities. Difficult mathematical problems cause him some anxiety, but if he had to solve a difficult problem he would overcome his anxiety and would do his best to solve it. When he faces difficulties during the solving of a problem, he revisits the problem and the given information and tries to connect them in a way that will lead him to the solution.

The management of problem solving in his classroom
The school teaching program, as he claims, does not offer many opportunities for problem solving activities. The time that should be spent on a problem depends on two factors, the
degree of the problem’s difficulty and the students’ abilities. A teacher should be flexible as regards time because he has to avoid creating the impression that “the fastest problem solver wins”. A whole teaching period (40 minutes) should not be spent on a problem, unless there is one and only objective for that lesson. “A problem that requires an entire teaching period in order to be solved must be very demanding and students will have to use advanced mental processes”. His students do not like mathematical problems and they prefer simple procedural exercises with only one particular answer and one particular way of arriving at it. When his students face difficulties during the process of solving a problem, he gives some hints so that the student can find their own way towards the solution. If he observes that some difficulties are common among the students, then he gives them group feedback.

Solving the problem
Mr Orestis was very calm when he was given the problem. Nonetheless, during solving the problem, he was methodical from the beginning. He identified all the pairs of numbers which could be the dimensions of triangles satisfying the three criteria. After drawing all the 12 triangles of the group “base 2, height 6”, he generalised his idea and claimed that there were 36 triangles for the problem.

The lesson based on the problem
During the lesson based on the problem, Mr Orestis offered hints to his students on how they could proceed with the solution. However, a big part of his lesson, especially at the end of it, was mostly based on the teacher’s demonstration of the solution. Selected incidents that justify the use of particular mathematical didactics are presented below.

Coaching
(a) The classroom was discussing the criterion of the problem which stated that all vertices should be on grid points. In Greek, there is no single word for “grid point”, so the Greek translation of the problem referred to this as the “points where the horizontal and vertical lines of the paper intercept”. The teacher asked the students if they understood what was meant. Some children argued that they were not sure. Then Mr Orestis drew the following shape on the board and asked the children if a vertex could be like that.

A student explained to her classmates that this could not happen. All children agreed.
(b) Mr Orestis asked the children: “How are we going to use the first piece of information that the problem gives us? 9 square units. Definitely, we are not going to draw triangles by chance”.

Explaining
(a) A student asked if a triangle with base 4 and height 4.5 could be a solution. The teacher told him that only integers should be used, without offering any further explanations.
(b) At the end of the lesson, the teacher told the students that they should look at all the cases of triangles with certain dimensions. He chose to demonstrate the triangles with base 6 and height 3, since he had already presented some of those cases earlier. When the bell rang, Mr Orestis told the children that if anyone was interested in finding all the solutions, he/she should try at home.
Conclusions
Many researchers have examined the relationship between beliefs and mathematics achievement, and specifically the attainment regarding problem solving. However, as already noted, the foci of their studies were on students and not teachers. For instance, the study of Nicolaidou and Philippou (2003) has shown that there is a significant correlation between Cypriot primary students’ attitudes towards mathematics, self-efficacy beliefs and performance on problem solving activities. Similarly, Mason’s (2003) study, concerning Italian students, revealed that the strongest predictors for mathematics achievement were two problem solving beliefs (belief regarding perceived ability to solve time-consuming problems and belief that not all problems can be solved by applying step-by-step procedures). We assert, however, that the connection between teachers’ beliefs about their problem solving competence and the observed competence is complicated and cannot be presented as a linear function.

The findings of the research reveal the complexity of the relation between teachers’ problem solving beliefs and competence. Moreover, they show that the interrelationships of beliefs and competence on teachers’ instructional practice are also complex with no simple cause and effect in much the same way as Thompson (1984) found with her three teachers. In closing we refer to Thompson’s appeal for teachers (1) to experience mathematical problem solving from the perspective of the problem solver before they can adequately deal with its teaching, (2) to reflect upon the thought processes that they use in solving problems to gain insights into the nature of the activity and (3) to become acquainted with the literature on research on problem solving and instruction in problem solving. According to Cooney (1985), studies suggested that teachers may not possess rich enough constructs to envision anything other than limited curricular objectives or teaching styles and hence may be handicapped in realising a problem solving orientation. The use of a problem-solving approach demands not only extensive preparation but also the development of ways to maintain at least a modicum of classroom control and, perhaps most importantly, the ability to envision goals of mathematics teaching in light of such an orientation.

References


Strategies for Solving Word Problems on Speed: A Comparative Study between Chinese and Singapore Students

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Abstract: The study was conducted to investigate strategies that Chinese and Singapore students used for solving word problems on speed. A test comprising 14 word problems on speed was administered to 1002 Singapore and 1070 Chinese students from Primary 6 to Secondary 2. A two-way country×grade ANOVA revealed that there was a significant interaction. The strategy analysis indicated that the Chinese students performed better than the Singapore students because they used algebraic strategies more frequently than the Singapore students. The Singapore students performed better than the Chinese students on one problem because the Primary-6 Singapore students successfully used model drawing and unitary methods. The Singapore students were found to use model drawing, unitary, guess-and-check, etc. more frequently than the Chinese students. However, the success rates of the use of these strategies were lower than those of the algebraic strategies that were used more frequently by the Chinese students. The study has some implications for the teaching and learning of speed, algebra, and problem solving in schools.

Keywords: Word problems, Speed, Cross-national comparison, Problem-solving strategies

Introduction

Problem solving has been included in Singapore syllabi since 1992 (Ministry of Education (MOE) (Singapore), 1990a, 1990b). It is explicitly postulated that “the primary aim of the mathematics programme is to enable pupils to develop their ability in mathematical problem solving” (MOE (Singapore), 2001a, 2001b, p.5). About twelve heuristics such as “act it out”, “draw a model/diagram” etc. are suggested in mathematics syllabi (MOE (Singapore), 2001a, 2001b). However, in China, the teaching of mathematics put more emphases on the learning of basic knowledge and the training of basic skills, also known as “Two Basics” (Zhang, Li, & Tang, 2004). Problem solving is taught after the teaching of basic mathematical concepts and techniques to illustrate their applications to the real world. The new curriculum standard (BNUP (China), 2001) has included problem solving as one of the four aspects of mathematics teaching and learning. A comparative study between the two countries is meaningful for getting insight into what are the advantages and disadvantages of the different ways of teaching problem solving in schools.

Word problems on speed are selected because they are application problems of various mathematical concepts from primary to university levels. The mathematics of change and variation, and in particular, the study of motion, is a fundamental concept
that underlies elementary algebra and calculus instruction (Bowers & Nickerson, 2000). Speed has been studied as conceptual problems (Acredolo & Schmid, 1981; Piaget, 1970; Zhou, Peverly, Boehm, & Lin, 2000; Zhou, Peverly, & Lin, 2004) with children at the age of 5 to 12 years. However, very little work has been done with word problems on speed as those included in textbooks. Though several studies include rate problems as a specific model of multiplication and division (Bell, Fischbein, & Greer, 1984; Fischbein, Deri, Nello, & Marino, 1985; Greer, 1992), the word problems on speed included in these studies are only the simplest one among the 13 categories of motion (speed) problems Mayer (1981) identified. Mayer analyzed word problems on speed in secondary school mathematics textbooks, but Mayer did not investigate how students actually solve the problems and what difficulties they may have. This study seeks in part to fill these gaps. Another reason is that word problems on speed in secondary school mathematics textbooks in China (Jiang, 1998a, 1998b; People’s Education Press (PEP), 1992, 1993a, 1993b, 1994) and in Singapore (MOE (Singapore), 2000a, 2000b; Teh & Looi, 2002a, 2002b).

The comparative study was conducted to answer the research question “Being taught various problem solving heuristics, do Singapore students perform better than Chinese students in solving word problems on speed?”

A problem-solving strategy model (Table 1) including nine strategies was developed from a review of textbooks and syllabi in the two countries and the Concepts in Secondary Mathematics and Science (CSMS) project (Hart, 1981). Here, strategies refer to the methods or problem solving procedures that direct the search for a solution (Krulik & Rudnick, 1988). The problem-solving strategy model was used to identify strategies used by the students.

Table 1

<table>
<thead>
<tr>
<th>No.</th>
<th>Strategy category</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Arithmetic method</td>
<td>It is used where the subject writes down a mathematical statement involving one or more operations on the numbers given in the problem (Fong &amp; Hsui, 1999).</td>
</tr>
<tr>
<td>2</td>
<td>Algebraic method</td>
<td>It is used when one or more unknowns are chosen as variables and equation(s) is set up.</td>
</tr>
<tr>
<td>3</td>
<td>Model drawing method</td>
<td>It is used when the solution is suggested by or follows a model or a diagram (Kho, 1987).</td>
</tr>
<tr>
<td>4</td>
<td>Guess-and-check</td>
<td>It involves the following two steps:</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(a) Make a guess of a certain answer or the unknown in the problem based on an estimation;</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(b) Check if the constraints given in the question or implied from some of the question statements are satisfied. If all the constraints are satisfied, the guess is correct; the</td>
</tr>
</tbody>
</table>
answer has been obtained or can be worked out. All processes will end at this point. If the constraints are not satisfied, the guess will be refined or adjusted, and another guess will be made, then another round of guess-and-check will begin.

5 Looking for a pattern It involves the following three steps:
(a) Several specific instances/special cases/particular examples of a problem are explored and listed;
(b) A pattern (or a conjecture, a generalization, a hypothesis, or common property among those special cases) is determined by investigating the special cases explored in Step (a); and
(c) A solution to the entire problem is found by applying the generalized result obtained in Step (b).

6 Unitary method Unitary method involves finding the value equivalent to one unit of a quantity from an equating statement and obtaining the value equivalent to more units of the quantities using the value for one unit just found (Fong, 1999; Fong & Hsu, 1999; Yuen, 1995).

7 Proportion method A proportion method is used when proportional properties (direct and inverse proportions) are used.

8 Logical reasoning Logical reasoning strategy is used when some forms of “if-then” reasoning are used (van De Walle, 1993).

9 No strategy It refers to the absence of a written response and where only pieces of information taken from the question are written down but without any continuing working (Fong & Hsu, 1999).

Method

1070 Chinese students (361 Primary 6, 354 Secondary 1, & 355 Secondary 2) and 1002 Singapore students (345 Primary 6, 315 Secondary 1, & 342 Secondary 2) participated in the study. The Chinese sample was from Wuhan City, China. A test was developed from an analysis of various types of word problems on speed (Jiang, 2005) and administered in intact classes. No calculators were allowed. Prior to the test, all the students have learned and completed the topic on speed.

Both quantitative and qualitative analyses were conducted. The responses were scored using a 0-1-2 scale. Two points were given to each correct answer or an incorrect answer where all the necessary steps are included but with only minor computational errors. One point was given to each answer that solved part of the
A zero ‘0’ point was given to answers that were completely wrong and to cases with no solution offered. The problem-solving strategy model was used to identify students’ responses to 11 problems where workings were required.

**Results**

This section has three parts. The first part is about the performance comparison results. The second part is trying to explain the performance differences from analyzing the strategies the students used. The third part presents detailed results about the use of strategies for solving three problems.

**Performance Comparison Results**

Table 2 shows the mean scores and standard deviations of the Chinese and the Singapore students in the test. A two-way country × grade ANOVA revealed that there was a significant interaction (F(2, 2066) = 22.88, \( p < .001 \)). Post hoc pairwise comparisons indicate that there were no statistically significant differences between the Chinese students at any two of the three grade levels. However, there were statistically significant differences between the Primary-6 and Secondary-1 Singapore students (Mean difference (MD) = 2.76, \( p < 0.001 \)) and between the Primary-6 and Secondary-2 Singapore students (MD = 3.38, \( p < 0.001 \)). There were no significant differences between the Secondary-1 and Secondary-2 Singapore students (MD = 0.62, \( p = 0.37 \)). The post hoc pairwise comparisons also showed that there were statistically significant differences between the Chinese and the Singapore students at each of the three grade levels (\( p < 0.001 \)). Therefore, the main difference existed between the Primary-6 and secondary Singapore students.

**Table 2**  
*Means and Standard Deviations of the Chinese and Singapore Students on the Test*

<table>
<thead>
<tr>
<th>Grade</th>
<th>China</th>
<th>Singapore</th>
<th>Mean Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>M</td>
<td>SD</td>
<td>M</td>
</tr>
<tr>
<td>Primary 6</td>
<td>21.95</td>
<td>5.41</td>
<td>19.52</td>
</tr>
<tr>
<td>Secondary 1</td>
<td>21.74</td>
<td>5.10</td>
<td>16.76</td>
</tr>
<tr>
<td>Secondary 2</td>
<td>22.20</td>
<td>5.05</td>
<td>16.13</td>
</tr>
<tr>
<td>Overall</td>
<td>21.96</td>
<td>5.19</td>
<td>17.49</td>
</tr>
</tbody>
</table>

**Differences in the Use of Strategies**

Comparisons in the use of strategies between different groups of students are made in terms of Strategy Percents (SP), which is the percent of a specific group of students using a specific category of strategy. Table 3 shows the SPs of the students for solving 11 problems.

**Table 3**  
*Strategy Percents (SP) of the Chinese and Singapore Students in Solving Problems in the Study*

<table>
<thead>
<tr>
<th>Strategy category</th>
<th>China</th>
<th>Singapore</th>
</tr>
</thead>
</table>

135
Data in Table 3 show that:

1. The students from both countries used the arithmetic strategies in most of the cases.

2. For the Chinese students, the second most frequently used strategy is algebraic strategies. For the Singapore students, the second most frequently used strategy is model drawing methods followed by unitary and guess-and-check methods.

3. The use of other three kinds of strategies (i.e., logical reasoning, proportion method, and looking for a pattern) was very rare.

4. The Singapore students had no strategy in more of the cases than their Chinese peers.

5. For the samples from the same country, the trends were different. For the Chinese sample, the secondary students used algebraic strategies more frequently than the arithmetic strategies. Though the Secondary-2 Singapore students did use the algebraic strategies more frequently than the Primary-6 and Secondary-1 Singapore students, they used the algebraic strategies even much less frequently than the Primary-6 Chinese students. The secondary Singapore students still used the arithmetic, model drawing, unitary, and guess-and-check methods as the Primary-6 Singapore students.

Therefore, the strategy analyses generally show that (a) the Singapore students used a greater variety of strategies for solving word problems on speed; and (b) the Chinese students used algebraic strategies more frequently than the Singapore students, especially the secondary Singapore students.

**Strategies for Solving Three Problems**

As examples, this part will show the detailed results about the use of strategies for solving two problems (Appendix A). Problem 1 is a typical algebraic word problem like Cows and Chickens Problem in Kaur (1998). Problem 2 describes a round trip. Knowledge of inverse proportion could be used for solving it. Problem 3 involves

<table>
<thead>
<tr>
<th>Strategy</th>
<th>P-6 (n=361)</th>
<th>S-1 (n=354)</th>
<th>S-2 (n=355)</th>
<th>Total (n=1070)</th>
<th>P-6 (n=345)</th>
<th>S-1 (n=315)</th>
<th>S-2 (n=342)</th>
<th>Total (n=1002)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Arithmetic method</td>
<td>68.24</td>
<td>38.50</td>
<td>42.51</td>
<td>49.86</td>
<td>47.01</td>
<td>0.26</td>
<td>4.89</td>
<td>1.81</td>
</tr>
<tr>
<td>Algebraic method</td>
<td>15.66</td>
<td>50.13</td>
<td>48.14</td>
<td>37.84</td>
<td>0.18</td>
<td>0.38</td>
<td>0.19</td>
<td>0.36</td>
</tr>
<tr>
<td>Model drawing method</td>
<td>0.13</td>
<td>0</td>
<td>0.05</td>
<td>0.06</td>
<td>20.26</td>
<td>0.63</td>
<td>0.35</td>
<td>0.48</td>
</tr>
<tr>
<td>Unitary method</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>12.02</td>
<td>9.90</td>
<td>10.55</td>
<td>10.85</td>
<td></td>
</tr>
<tr>
<td>Guess-and-check</td>
<td>2.37</td>
<td>0.15</td>
<td>0.18</td>
<td>0.91</td>
<td>13.18</td>
<td>8.60</td>
<td>5.21</td>
<td>9.02</td>
</tr>
<tr>
<td>Logical reasoning</td>
<td>2.04</td>
<td>0.10</td>
<td>0.20</td>
<td>0.79</td>
<td>0.47</td>
<td>0.63</td>
<td>0.35</td>
<td>0.48</td>
</tr>
<tr>
<td>Proportion method</td>
<td>1.44</td>
<td>0.46</td>
<td>0.18</td>
<td>0.70</td>
<td>0.53</td>
<td>0.38</td>
<td>0.19</td>
<td>0.36</td>
</tr>
<tr>
<td>Looking for a pattern</td>
<td>0.05</td>
<td>0</td>
<td>0</td>
<td>0.02</td>
<td>0.34</td>
<td>0.23</td>
<td>0.13</td>
<td>0.24</td>
</tr>
<tr>
<td>No strategy</td>
<td>10.07</td>
<td>10.66</td>
<td>8.73</td>
<td>9.82</td>
<td>6.01</td>
<td>15.44</td>
<td>18.61</td>
<td>13.27</td>
</tr>
</tbody>
</table>

Note. P-6 = Primary 6; S-1 = Secondary 1; S-2 = Secondary 2.

*The numbers are the percentages (% is omitted) of the cases (No. of problems (11) × No. of students) where the group of students used the specific kind of strategies.
fractions to represent the relationships among the distances. This kind of problems were found in a popular workbook written by Fong (1998).

For solving individual problems, we do not only use SP for comparison, but also use Success Rate (SR). SR is the proportion of the specific group who could use the strategy to get the correct answers to a problem. It is used to measure how successfully the strategy is used by different group of students. If the Success Rate Difference (SRD) between different groups of students is more than 10%, it is taken as high. If the success rate of a strategy is lower than 30%, the strategy is taken as inappropriate for solving a specific problem because a majority (70%) of the students using the strategy could not reach the correct answer. Table 4 shows the SPs and SRs in the use of strategies for solving the three problems.

Table 4
*Strategy Percents (SP) and Success Rates (SR) of Chinese and Singapore Students in Solving the Three Problems*

<table>
<thead>
<tr>
<th>Strategies</th>
<th>China</th>
<th>Singapore</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>P-6 (n=361)</td>
<td>S-1 (n=354)</td>
</tr>
<tr>
<td>-------------------</td>
<td>----------------</td>
<td>-----------------</td>
</tr>
<tr>
<td><strong>Problem 1</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Arithmetic Method</td>
<td>8.31</td>
<td>3.11</td>
</tr>
<tr>
<td>SR</td>
<td>6.67</td>
<td>0</td>
</tr>
<tr>
<td>Algebraic Method</td>
<td>46.54</td>
<td>82.20</td>
</tr>
<tr>
<td>SR</td>
<td>92.26</td>
<td>91.75</td>
</tr>
<tr>
<td>Guess-and-check</td>
<td>18.84</td>
<td>1.69</td>
</tr>
<tr>
<td>SR</td>
<td>97.06</td>
<td>100</td>
</tr>
<tr>
<td>Logical Reasoning</td>
<td>12.19</td>
<td>0.56</td>
</tr>
<tr>
<td>SR</td>
<td>75.00</td>
<td>100</td>
</tr>
<tr>
<td>Model drawing</td>
<td>0.28</td>
<td>--</td>
</tr>
<tr>
<td>SR</td>
<td>100.00</td>
<td>--</td>
</tr>
<tr>
<td>Looking for a pattern</td>
<td>--</td>
<td>--</td>
</tr>
<tr>
<td>SR</td>
<td>--</td>
<td>--</td>
</tr>
<tr>
<td>No strategy</td>
<td>13.85</td>
<td>12.43</td>
</tr>
</tbody>
</table>

| **Problem 2**     |                |                 |             |               |                |                 |             |               |
| Arithmetic method | 30.75          | 22.88           | 21.97       | 25.33         | 54.20          | 46.35           | 43.27       | 48.00          |
| SR                | 0              | 3.66            | 0           | 1.11          | 0              | 0.68            | 0           | 0.21           |
| Algebraic method  | 31.86          | 53.39           | 62.82       | 49.25         | --             | 0.32            | 11.70       | 4.09           |
| SR                | 87.83          | 67.20           | 63.23       | 70.02         | --             | 100.00          | 65.00       | 65.85          |
| Guess-and-check   | 2.77           | --              | --          | 0.93          | 21.16          | 11.11           | 6.43        | 12.97          |
| SR                | 80.00          | --              | --          | 80.00         | 90.41          | 74.29           | 72.73       | 83.08          |
| Proportion method | 15.79          | 5.08            | 1.97        | 7.66          | 5.22           | 3.81            | 1.46        | 3.49           |
| SR                | 82.46          | 94.44           | 100.00      | 86.59         | 83.33          | 75.00           | 40.00       | 71.43          |
| Model             | --             | --              | --          | --            | 7.83           | 3.81            | 2.05        | 4.59           |
For Problem 1, we can have the following observations:

1. A higher percentage of the Singapore students used the inappropriate arithmetic strategies (SR < 5%) than that of the Chinese students (SPD = 21%). It is surprising to find that the higher the grade, the higher percentage of the Singapore students used this inappropriate strategies.

2. A majority (70%) of the Chinese students used the effective algebraic strategies (SR > 80%). However, only about 5% of the Singapore students did so. The Chinese students used the algebraic strategies better than their Singapore peers (SRD > 10%).

3. Up to 45% of the Singapore students used the guess-and-check strategies. However, only 7% of the Chinese students did so. The students from the two countries used the guess-and-check strategies equally successfully (SRD = 2%).

4. About 5% of the Chinese students and 4% of the Singapore students used logical reasoning method. The Chinese students used the logical reasoning method more successfully than their Singapore peers (SRD = 10%).

5. A higher percentage of the Singapore students used the inappropriate model drawing methods (SR < 30%) than the Chinese student (SPD = 5%).

6. One Singapore student was found to use the “looking for a pattern” method to solve this problem.

7. Similar percentages of the Singapore students and the Chinese students had no strategies (SPD = 2%). However, for the students at the three grade levels, the trends are different. For Primary-6 samples, a higher percentage of the Chinese students had no strategies than their Singapore peers (SPD = 9%). However, for secondary samples, higher percentages of the Singapore students had no strategies
than those of their Chinese peers (SPD = 6-9%).

For Problem 2, we can have the following observations:

(1) A higher percentage of the Singapore students used the inappropriate arithmetic method (SR < 2%) than that of the Chinese students (SPD = 23%). Most of them used irrelevant procedures like “(120+40)÷2 = 80”, “120+40 = 160, 160×2 = 320”, etc.

(2) A higher percentage of the Chinese students used the appropriate algebraic strategies (SR > 65%) than that of the Singapore students (SPD = 45%). The students from both countries could use the algebraic strategies equally well (SRD = 4%).

(3) A higher percentage of the Singapore students used the guess-and-check methods than that of the Chinese students (SPD = 12%). The Primary-6 Singapore students used this method more successfully than the secondary Singapore students.

(4) A higher percentage of the Chinese students used the proportion methods than that of the Singapore students (SPD = 5%). The Chinese students used the proportion methods more successfully than the Singapore students (SRD = 15%). For the Chinese students, the higher the grade, the more successfully they used proportion method. However, for the Singapore students, it was reversed.

(5) About 6% of the Singapore students used the appropriate model drawing (SR = 37%) and unitary methods (SR = 65%). However, no Chinese students were found to use these two kinds of methods.

(6) Very few students from the two countries used logical reasoning strategies (CN 2%; SG 1.3%) though their success rates are high.

(7) A higher percentage of the Singapore students had no strategies than that of their Chinese peers (SPD = 9%).

For Problem 3, we can have the following observations:

(1) A majority (55%) of the Chinese students used arithmetic strategies, especially for the Primary-6 Chinese students. However, only 10% of the Singapore students used this method. The Chinese students used this method more successfully than the Singapore students (SRD = 30%).

(2) About 36% of the Chinese students used algebraic strategies. Higher percentages of the secondary Chinese students used this method than that of the Primary-6 Chinese students (SPD > 40%). However, very few Singapore students were found to use algebraic method. The secondary Chinese students used this method more successfully than the Primary-6 Chinese students (SRD > 15%).

(3) About half of the Singapore students used model drawing method, but very few Chinese students did so. The success rate of this method of the Singapore students is not low (SR = 54%), especially for the Primary-6 students (SR = 66%). However, the secondary Singapore students used this method much less successfully than the Primary-6 Singapore students (SRD > 18%).

(4) Up to 40% of the Singapore students used unitary methods for solving this problem. The secondary Singapore students used this method much less
successfully than the Primary-6 Singapore students (SRD > 11%).

(5) A lower percentage of the Singapore students had no strategies than that of the Chinese students (SPD = 6%). This is also true for the samples at each of the three grade levels.

In conclusion, (a) the Singapore students used the inappropriate arithmetic strategies more frequently than their Chinese peers; (b) the Chinese students used algebraic strategies more frequently and more successfully than their Singapore peers; (c) the Singapore students used more varied strategies including guess-and-check, model drawing and unitary methods, etc. However, the secondary Singapore students could not use these strategies as successfully as the Primary-6 Singapore students.

**Summary and Discussion**

Cross-national studies provide us with an opportunity to ascertain the strength and weakness of educational systems (Robitaille & Travels, 1992), and consequently provide information about how to improve teaching and learning of mathematics (Cai, 2000, 2004; Robitaille & Travers, 1992). This comparative study has severeral implications for the teaching and learning of mathematics in schools.

First, the teaching of various heuristics in Singapore schools has equipped the Singapore students with more ways to tackle problems. This kind of practice has also made the Primary-6 Singapore students have no strategy in a lower percentage of the cases than their Chinese peers. The use of model drawing and unitary methods has converted the multiplication and division of fractions (Problem 3) to the multiplication and division of whole numbers. This kind of conversion might have removed the difficulties that the Chinese students experienced. This finding may explain the Singapore students’ good performance in number and proportionality in the TIMSS studies. In the TIMSS studies, the secondary Singapore students performed the best in number and proportionality (Beaton, et al., 1996; Mullis, et al., 2000; Mullis, et al., 2004). Therefore, it is suggested to teach the Chinese students the problem-solving heuristics that are proved to be effective, such as guess-and-check, model drawing and unitary methods.

Second, the strategy analyses show that the secondary Singapore students do not use algebraic strategies as frequently as the Chinese students. Problem 1 is similar to the Cows and Chickens Problem in the study of Kaur (1998) and the Correct Answer Problem in the study of Loh (1991). The results obtained in the study are consistent with the studies of Kaur and Loh in that (a) few Secondary-2 Singapore students were found to use algebraic methods, (b) the Singapore students preferred to use guess-and-check methods and could use them efficiently, and (c) about one-third of them used irrelevant procedures. Consequently, effort needs to be made to help the secondary students recognize the strength of algebraic methods for solving word problems as well as to bridge the gap between primary and secondary mathematics (Fong, 1994).

Third, the secondary Singapore students used the arithmetic, model drawing, unitary, and guess-and-check methods less successfully than the Primary-6 Singapore
students. Therefore, effort needs to be made to help the secondary students consolidate what they have learnt in primary schools.

Appendix A
1. On Sunday, Judy went to see her grandma who lives 150 km apart. After cycling at an average speed of 15 km/h for a few hours, she got tired and took a lift from the passing truck. The truck’s average travelling speed is 75 km/h. When she got to her grandma's house, she checked the time and knew that the trip took her 6 hours. Find the time she cycled?

2. Sunday morning, Rebecca and her parents went out to enjoy the natural scenery. On the way to the destination, they travelled at a slow speed of 40 km/h. On the way back, they drove at a faster speed of 120 km/h. When they came back home, they found that they had been out for 2 hours. Find the average speed for this round trip (Ignoring time around the destination).

3. Mike made a journey from City P to City Q. In the first half an hour, he covered \( \frac{1}{7} \) of it. In the second half an hour he covered \( \frac{1}{5} \) of the remaining journey. Finally he took another half an hour to finish the journey at a speed of 72 km/h. Calculate his average speed for the whole journey.

Reference:


Ministry of Education (MOE) (Singapore). (1990a). *Mathematics syllabus: Primary 1 to 3, Primary 4 to 6 (Normal Course), Primary 4 to 8 (Extended Course).* Ministry of Education (Singapore): Curriculum Planning Division.


Beyond Show and Tell: *Neritage* for Teaching through Problem-Solving
- Ideas from Japanese Problem-Solving Approaches for Teaching Mathematics -

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DePaul University

Abstract

Japanese teachers use problem solving as a powerful approach for teaching mathematics. There are several notable characteristics of the Japanese approach to problem solving. One of the characteristics is that Japanese problem solving lessons usually do not end even after each student finds a solution to the problem. Japanese teachers and researchers believe that the heart of the lesson begins *after* students come up with solutions. The teacher facilitates extensive discussion with students, which is called *Neritage*, by comparing and highlighting the similarities and differences among students’ solutions. In this paper, some of the characteristics of the Japanese approach for teaching mathematics will be discussed by focusing on the heart of the approach, *Neritage*.
Introduction

Problem solving has been a major focus in Japanese mathematics curricula for nearly a half century. Numerous teacher reference books and lesson plans using problem solving have been published since the 1960s. Government authorized mathematics textbooks for elementary grades, which are published by six private companies, have had more and more problem solving over the years. As a result, almost every chapter in recent Japanese mathematics textbooks for elementary grades begins with problem solving as a way to introduce new concepts and ideas to students.

A few key publications have greatly influenced how problem solving is used in Japanese mathematics education. Polya’s *How to Solve It* (Polya, 1945, p. 26) was translated and published in Japanese in 1954, and had been studied by various researchers and educators in Japan. Japanese researchers, teachers, and administrators worked collaboratively through Lesson Study, a professional development approach that is popular in Japan, to develop mathematics instruction by referring to Polya’s (1945) four phases of problem solving work (Takahashi, 2000). One of the results from the studies of problem solving, *Open-ended Approach*, was published in 1977 by Shimada et al. The open-ended approach has been widely used in Japanese classrooms since then. Moreover, the English translation of the book was published (J. P. Becker & Shimada, 1997) and has been popular among educators in the U.S. too. The Ministry of Education in Japan has recognized the importance of problem solving in school mathematics and emphasized the need for students to develop problem-solving skills to learn and use mathematics in various documents since the beginning of the 1980s. The position statement from the NCTM's *An agenda for action: Recommendations for school mathematics of the 1980s* (1980) that "problem solving must be the focus of school mathematics" was referenced in various research articles and resource materials for teachers in Japan during the 1980s. Also, *Teaching Problem Solving: What, why & how* (Charles & Lester, 1982) was translated into Japanese in 1983.

Stigler and Hiebert (1999) described Japanese mathematics lessons as “structured problem solving”. Similar characteristics of Japanese mathematics lessons were also reported in the proceedings of the U.S.-Japan Seminar of Mathematical Problem Solving (Jerry P. Becker & Miwa, 1987; Jerry P Becker, Silver, Kantowski, Travers, & Wilson, 1990). Structured problem solving is designed for students to acquire knowledge and skills through creative mathematical activity by presenting challenging problems to students. Students are expected to solve a problem
using their own mathematical knowledge. Thus, Japanese teachers usually do not tell students how to solve a problem before students try to solve the problem by themselves. Working with problems, students bring several different approaches and solutions to the class. The teacher then leads students in a whole-class discussion in order to compare individual approaches and solutions. This whole-class activity provides students with opportunities to learn mathematics. Through their extensive study of problem solving, Japanese teachers and educators have come to recognize that this whole-class discussion is the heart of structured problem solving and have named this discussion part *Neriage*.

In this paper, I will discuss the Japanese approach of using problem solving for teaching mathematics, structured problem solving, by focusing on the heart of the approach, *Neriage*.

The Japanese Problem Solving Approach

One of the major goals of teaching mathematics is to help students become able to solve problems. Thus, mathematics lessons employing problem solving are sometimes viewed as an approach for students to develop problem-solving skills and strategies, and teachers sometimes focus solely on the strategy of solving the problem and not necessarily on developing mathematical concepts and skills. This interpretation of problem solving lessons usually ends after each student comes up with a solution to the problem. The teachers’ role during students’ problem solving is to help students find the solution by providing an efficient strategy, because the major goal of the lesson is for students to solve problems.

On the other hand, problem solving can also be viewed as a powerful approach for developing mathematical concepts and skills. Thus, in this approach teachers use problem solving not only for lessons that solely focus on developing problem-solving skills and strategies but also on lessons that develop mathematical concepts, skills, and procedures. As a result the lesson plans for this approach usually include content goals in addition to goals for developing problem solving strategies and skills.

To highlight the difference between the two approaches, the latter approach is often called "teaching through problem solving." Since Japanese structured problem solving uses problem solving as a process for learning mathematical content, it could be considered a type of teaching through problem solving. Looking at problem solving as a process for students to learn mathematics is not unique to Japanese mathematics education. The National Council of Teachers of Mathematics (NCTM) has also emphasized the importance of teaching mathematics through
problem solving (Mathematics, 2006; National Council of Teachers of Mathematics, 1989, 2000, 2006). These documents discuss the necessity of learning mathematical content through the processes of problem solving, reasoning and proof, communication, connections, and representation. Various reform documents have also suggested that mathematics lessons should be designed to provide students learning opportunities through the processes of problem solving and not simply by listening to teachers’ lectures.

Although teaching through problem solving has been suggested by NCTM and other reform documents in the U.S., it is hard to find lessons that employ this idea in U.S. classrooms. Stigler and Hiebert argue that Japanese mathematics lessons better exemplify current U.S. reform ideas, such as teaching through problem solving, than do typical U.S. mathematics lessons, based on the TIMSS videotape classroom study (1997). One of the reasons behind this phenomenon might be that Japanese mathematics teaching already had a history of focusing on developing mathematical thinking skills by using a variety of story problems even before the idea of problem solving was introduced. Therefore Japanese educators looked at problem solving as an ideal approach for learning mathematics rather than simply a way to promote problem solving skills when the idea of problem solving was introduced.

There are several notable characteristics of the Japanese problem solving approach. First, the Japanese problem solving approach can be found throughout the curriculum because it is designed for learning mathematical content. Japanese teachers have tried to use the approach for students not only to develop concepts and understanding of mathematics but also to acquire skills to learn and use mathematics. Therefore problem solving is not viewed as an end-of-the-chapter activity that is solely focused on developing problem-solving skills and strategies. Second, Japanese problem solving lessons usually do not end even after each student finds a solution to the problem. Problem solving lessons that solely focus on developing problem solving and skills often end after students share their solutions with the class. Japanese teachers and researchers, however, believe that the heart of the lesson begins after each student comes up with a solution(s). Japanese teachers facilitate extensive discussion with students, which is called Neriage, by comparing and highlighting the similarities and differences among students’ solution approaches.
Extensive discussion (Neriage)

The term Neriage has been widely used among Japanese teachers and researchers of mathematics education as a technical term since 1980s. Neriage is a noun form of a verb Neriageru, which means to polish up. The term Neriage is used among Japanese teachers for describing the dynamic and collaborative nature of a whole-class discussion in the lesson (Shimizu, 1999). The most important role of the teacher in Neriage is to orchestrate students’ ideas and approaches to solve the problem and to help them polish their solutions in order to learn mathematical content. During the process, a teacher highlights important mathematical ideas and concepts for students to reach the goals of the lesson. This is why Japanese teachers see Neriage as the heart of teaching mathematics through problem solving. From the viewpoint of Japanese teachers, the solving of the problem by each student at the beginning of the lesson is a preparation for Neriage. Therefore it is important for students to struggle with the problem and find their own way to solve the problem, because this experience will be the foundation for students to make a connection between their previous learning and the content that they are going to learn through Neriage.

The following is an example of a series of problem solving activities from the most widely used Japanese mathematics textbook series. This example shows how Neriage leads students to acquire a new idea in mathematics through a series of problem solving activities. The
unit on Per Unit Quantity in the 5th grade textbook begins with the following problem (Figure 1) without showing any solution or hint to the students on the page. The textbook is designed for students to see everyday situations as mathematical problems. Most 5th grade students may realize that the crowdedness of Cabin A and Cabin B may be compared by looking at the number of the people who share each cabin and because the areas of the rooms look the same. Students may also be able to see that the crowdedness of Cabin B and Cabin C may be compared by looking at their area because the number of the people who share each cabin are the same. The textbook expects teachers to facilitate the above discussion at the beginning of the lesson so that students understand the situation. Then, the next page of the textbook (Table 2) provides further information for students to understand what is the mathematical problem in this everyday situation.

<table>
<thead>
<tr>
<th></th>
<th>Area (m²)</th>
<th>Number of people</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cabin A</td>
<td>16</td>
<td>6</td>
</tr>
<tr>
<td>Cabin B</td>
<td>16</td>
<td>5</td>
</tr>
<tr>
<td>Cabin C</td>
<td>15</td>
<td>5</td>
</tr>
</tbody>
</table>

Table 2. Which cabin is the most crowded?

By looking at the data, students are able to compare the crowdedness of Cabin A and Cabin B without any calculation. They also may be able to see which room is more crowded between Cabin B and Cabin C. It is, however, not easy for students to figure out which is more crowded between Cabin A and Cabin C. Through this discussion, teachers are expected to lead students to understand what is the problem to be solved. There are two data for each cabin, the area of the cabin and the number of people who share the cabin. It is easy to compare the crowdedness of the cabins if one of these two data are the same. Since neither the area nor the number of people for Cabin A and Cabin C are the same, it might not be possible to compare the crowdedness by using the data. If so, are there any ways to make one of the data the same -- either the area or the number of the people?

After the above discussion, each student is encouraged to figure out which is more crowded, Cabin A and Cabin C. Since the numbers in the problem are carefully chosen, there are several approaches for students to compare crowdedness. The following are four commonly seen student solution methods.
Method A:
Cabin A: \( 6 \div 16 = 0.375 \)
Cabin B: \( 5 \div 15 = 0.33 \cdots \)
Division is used to find how many people occupy 1 \( \text{m}^2 \). Because a larger number of people would occupy 1 \( \text{m}^2 \), Cabin A is more crowded.

Method B:
Cabin A: \( 16 \div 6 = 2.66 \cdots \)
Cabin B: \( 15 \div 5 = 3 \)
Division is used to find how many square meters there are per person. Because there is less area per person, Cabin A is more crowded.

Method C:
A common multiple of 16 and 15 is 240
Cabin A: \( 6 \times 15 = 90 \)
Cabin B: \( 5 \times 16 = 80 \)
This method looks at how many people would share each cabin if both cabins have the same area. In order to use this method, a common multiple 240 is found as the area of each cabin. Because more people would share 240 \( \text{m}^2 \), Cabin A is more crowded.

Method D:
A common multiple of 6 and 5 is 30
Cabin A: \( 16 \times 5 = 80 \)
Cabin B: \( 15 \times 6 = 90 \)
This method looks at how much area would be shared by a person if both cabins have the same area. In order to use this method, a common multiple 30 has been found as the number of people in each cabin. Because less area would be occupied by a person, Cabin A is more crowded.

After students come up with solution methods that include the above four methods, the teacher assigns students to share their solutions. Japanese teachers usually monitor students’ work during their individual or group problem-solving time and come up with a plan for the discussion. For example, teachers often use a seating chart of the class to jot down how each student approaches the problem as well as to plan how to lead the discussion. A teacher might
ask one of the students who used the most common method to share his/her method to begin the discussion, and then ask another student to share different methods. At this time, Japanese teachers usually do not say that the answers are right or wrong in order to provide students with opportunity to think carefully about each solution method. Teachers carefully use blackboard writing for students to see all the solution methods from their peers and to help them understand each method.

The Neriage begins after the students present their various solution methods. Until Neriage begins, the whole class activity is very similar to the children's favorite school activity, Show and Tell. Neriage is an activity that goes beyond Show and Tell, however. If the goal of the problem-solving lesson is just to find a solution to the problem, this could be the end of the lesson. But because the Japanese problem-solving lesson is designed for students to learn new mathematical knowledge, the Neriage is necessary.

Teachers might begin the Neriage by asking students to see if there are some common ideas or approaches among the solution methods or some differences. For example, students might notice that both Method A and Method B use division but in different ways. In contrast Method C and Method D use multiplication instead of division. On the other hand, Method A and Method C use the same idea that looks at what if both cabins had the same area. Method B and Method C also share the same idea that looks at what if both cabins were shared by the same number of people. During the discussion, teachers would provide opportunities to think about why using addition and subtraction might not be the best approach to this problem. This comparison would help students deepen their understanding of the concept and the use of multiplication and division concerning ratio and rate. If there are some methods based on students’ misunderstanding, teachers can use them to help students develop reasoning skills to justify whether the solutions are right.

Then, teachers would lead students to see whether each approach has advantages and limitations. Students might realize that the methods that use a common multiple might have a limitation if the number in the problem becomes larger or if the situation requires them to compare more than two rooms. Finding a common multiple with large numbers or with several numbers might not be an easy task. Considering this limitation, teachers might be able to conclude that the use of division to find a unit quantity might be a better way for the similar problem with a more complex situation. In fact, the Japanese textbook series gives students the
following problem after “Which cabin is the most crowded?” for students to actually see which method, multiplication or division, is most useful.

At Yoshiko’s farm, which is 600m$^2$, 1968kg of potatoes were produced. At Tadashi’s farm, which is 900m$^2$, 2682kg of potatoes were produced. Which farm was better at producing potatoes? (Hironaka & Sugiyama, 2006)

Teachers would also lead students to see whether there are any advantages to Method A and Method B. Comparing these two methods requires students to think deeply about the meaning of division. Moreover, this opportunity can help students to understand practical application of the use of division, since the order of division requires students to use different interpretations of the quotients. The quotients in Method A show that the larger quotient indicates more crowdedness. The quotients in Method B show that the larger quotient indicates less crowdedness. To use a number to represent crowdedness, the quotients in Method A are more understandable, because the larger number means that it is more crowded. Later in the unit, this Japanese textbook introduces population density, comparing crowdedness by using the number of people who live in an area of 1km$^2$, and asking students to solve the problem.

*Neriage* is a critical component of a Japanese problem-solving lesson because this is the place where teachers can teach students new mathematical ideas and concept by using students’ solution methods. Because the *Neriage* is built upon the students’ solutions as a foundation of the

Figure 2. Population density
Reprinted from *Mathematics for Elementary School 5B*. Tokyo, Japan: Tokyo Shoseki Co., Ltd. p.27 (Hironaka & Sugiyama, 2006)

dynamic and collaborative whole-class discussion, Japanese teachers put so much effort to preparing the discussion.
Lesson Planning for Problem Solving Lesson: Beyond Show and Tell

At the beginning of the development of the Japanese problem solving approach, the term *Neriage* had not been widely used. Instead, educators used the Japanese translation of terms from Polya’s four phases of problem solving—understanding the problem, devising a plan, carrying out the plan, and looking back—as a framework to design lessons. For example, 1) the problem solving lesson usually began with the presentation of the problem of the day by the teacher and the teacher helping students to understand what the problem really is, thus, "understanding the problem", 2) then the teacher asked each student to devise a plan to solve the problem, thus "devise a plan", 3) next, students solved the problem by using their previously learned knowledge and skills, thus "to solve the problem based on the plan," and finally, 4) students examined whether their solution was correct and the method that they used was reasonable and efficient, thus, "looking back." Although Polya’s four phases of problem solving had been used as the foundation of Japanese problem solving lessons, Japanese teachers revised the framework over the years through lesson study. There are two notable changes over the years. First, the 2nd phase, devise a plan, was omitted from the flow of the lesson. Second, to describe the 4th phase, looking back, the term *Neriage* began to be used. These changes did not happen in a top-down manner. Through numerous research lessons and post-lesson discussions, teachers gradually shifted their use of problem solving and reached the problem solving approach which Stigler and Hiebert (1999) described as structured problem solving.

Teaching mathematics through problem solving is not an easy task for teachers, especially facilitating a good discussion, *Neriage*. To develop problem-solving lessons, Japanese teachers usually begin by considering the following three major issues—the curriculum, the students, and the problem. By investigating the curriculum teachers should be able to identify what content should be taught. Japanese teachers try to identify the contents as specific as possible so that the lesson will be focused on a specific topic(s). Then, they examine students’ previous learning to identify the goal of the problem-solving lesson and to identify the problem for the lesson. Japanese teachers seek meaningful problems for problem solving lessons by looking at Japanese mathematics textbooks and resources materials. These resource materials include lesson study reports and books published by experienced lesson study practitioners. These three issues can be discussed in any order.
Sometimes Japanese teachers develop lessons for lesson study from scratch but they often develop lessons by using others' work and modify them to fit into their own students’ needs.

One of the challenges is to find a good problem that can lead students to accomplish the goals of the lesson. There are a lot of interesting and engaging problems, including puzzles and games; however, the problem should be able to foster students' ability to learn something new after they have solved the problem by using their existing knowledge and skills. In other words, when students solve the problem it is expected that the problem provides students with opportunities for see a need for learning new knowledge and skills, which is the goal of the lesson.

A group of teachers carefully examine problems for students. They always solve each problem by themselves in several different ways in order to examine whether the problem is mathematically meaningful for the students at the time of the lesson. Then, teachers begin to anticipate students’ responses to the problem including ones based on misunderstandings and misuses of previous learning. Then they start to design the flow of the lesson so that students will be able to reach the goal of the lesson through problem-solving. Coming up with good questioning for students to think deeply about mathematics is always a challenge for teachers when designing lessons.

One of the major tasks for Japanese teachers is to facilitate meaningful mathematical discussion during the whole-class activity to help students to achieve the goals of the lesson. When a teacher presents a problem to students without giving a procedure, it is natural that several different approaches to the solution will come from the students. Thus, the textbooks include examples of students’ typical approaches and ideas. Because the goal of the structured problem-solving approach is to develop students’ understanding of mathematical concepts and skills, a teacher is expected to facilitate mathematical discussion for students to achieve this goal. This discussion is often called \textit{Neritage} in Japanese, which implies polishing ideas. In order to do this, teachers need a clear plan for the discussion as a part of their lesson plans, which will anticipate the variety of solution methods that their students might bring to the discussion. These anticipated solution methods include not only the most efficient methods but also ones caused by students’ misunderstandings. Thus, anticipating students’ solution methods is a major part of lesson planning for Japanese teachers.
Towards the end of a lesson, a teacher often leads the lesson in a way that pulls all the
different approaches and ideas together in order to see the connection. Then, he or she
summarizes the lesson to help students achieve the objective of the lesson. The teacher often asks
students to reflect on what they have learned during the lesson.

Japanese teachers and researchers believe that *Neriage* is the most important component
of teaching mathematics through problem solving. Therefore, teachers spend much time
investigating various resources through *Kyozaikenkyu* in lesson study (Watanabe, Takahashi, &
Yoshida, in press). The quality of *Neriage* depends upon how well the teacher(s) plan the lesson,
because this is the place where teachers have to use all their knowledge of mathematics, their
knowledge about teaching mathematics, their knowledge of students, and their skills to facilitate
the whole-class discussion. Therefore, Japanese educators believe that teaching, especially in
*Neriage*, is the proving ground of teachers’ knowledge and skills (Fujii, in press).

Conclusion

This paper focuses on the term *Neriage* to highlight the characteristics of a Japanese
teaching approach based on problem solving. There are several other technical terms in the
Japanese educational community. These technical terms are sometimes used among teachers to
describe and discuss specific events or techniques in teaching and learning. This means that a
term such as *Neriage* is used only among teachers, and people in other professions do not share
the meaning of the term. The existence of such technical terms tells us that Japanese teachers
have an opportunity to discuss teaching and learning with their colleagues regularly.

The Japanese teaching profession has a long history of collaboration among teachers to
discuss how to improve teaching and learning. The Japanese problem-solving approach is one of
the outcomes of this collaboration. Thus, the development of collegial communities among
teachers is an ideal way to make a shift in the teaching and learning of mathematics by using
suggestions from various documents.
References


Instructional Practices to Facilitate Prospective Mathematics Teachers’ Learning of Problem Solving for Teaching

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Introduction

Problem solving is considered central to school mathematics. It is highlighted in reform documents as a key factor of change in mathematics education (NCTM 1989, 1991, 2000). As NCTM (2000, p. 52) states,

Instructional programs should enable all students to build new mathematical knowledge through problem solving; solve problems that arise in mathematics and in other contexts; apply and adapt a variety of appropriate strategies to solve problems; and monitor and reflect on the process of mathematical problem solving.

Similarly, Kilpatrick, Swafford, and Findell (2001, p. 420) explained,

Studies in almost every domain of mathematics have demonstrated that problem solving provides an important context in which students can learn about number and other mathematical topics. Problem-solving ability is enhanced when students have opportunities to solve problems themselves and to see problems being solved. Further, problem solving can provide the site for learning new concepts and for practicing learned skills.

Thus, problem solving is important as a way of doing, learning and teaching mathematics. However, whether or how such ways of viewing problem solving get implemented in the classroom will depend on the teacher. In addition, Schoenfeld (1985) has identified four aspects of students’ problem solving that can be used to guide instruction – resources, heuristics, control, and beliefs. This means that teachers have to play a central role in helping students choose resources, implement heuristics, control their problem solving actions, and develop useful beliefs.

If problem solving should be taught to students, then it should be taught to prospective teachers who are likely to enter teacher preparation programs without having been taught it in an explicit way. If it is to form a basis of teaching mathematics, then prospective teachers should understand it from a pedagogical perspective. There are studies, discussed later, that raise issues about prospective teachers’ understanding of problem solving and ability as problem solvers that could affect what and how they implement problem solving in their teaching. It therefore seems to be important that teacher education includes learning opportunities explicitly focused on problem solving. In this paper, I draw on studies, including my own work, that include instructional practices to facilitate prospective teachers’ learning of problem solving and problem-solving pedagogy in order to highlight the nature of these practices and the learning that results from them and to discuss key characteristics of the practices that have implications for how we prepare prospective teachers to use problem solving in their teaching. The paper, then, is based on a review of research literature and a report of a study I conducted to identify (i) prospective mathematics teachers’ knowledge and ability of problem solving; (ii) instructional approaches to facilitate their learning of problem solving; and (iii) implications for teacher education.
Prospective Teachers’ Knowledge of Problem Solving

Studies focusing explicitly on prospective teachers’ knowledge of problem solving are a scarcity in the research literature, regardless of whether routine or non-routine problems are considered. What are available deals mainly with the prospective teachers’ ability or strategies in solving word problems, as in these examples. Schmidt and Bednarz (1995) examined modes of problem solving that 131 prospective elementary and secondary teachers used in arithmetical and algebraic word problems to identify the resistance and eventual difficulties that arose in the shift from one type of approach to the other. The majority of them confined themselves to algebra even when dealing with arithmetical problems. The prospective elementary teachers appeared to be the best prepared for addressing both fields, i.e., used arithmetic for arithmetic problems and algebra for algebra problems. van Dooren, Verschaffel, and Onghena (2003) investigated the arithmetic and algebraic word problem-solving skills and strategies of 97 prospective elementary and secondary teachers, both at the beginning and at the end of their teacher preparation. They found that the prospective secondary teachers clearly preferred algebra, even for solving very easy problems for which arithmetic was more appropriate. About half of the prospective elementary teachers adaptively switched between arithmetic and algebra, while the other half experienced serious difficulties with algebra. Contreras & Martínez-Cruz (2001) examined 68 prospective elementary teachers’ solution processes to a word problem involving division of fractions in which the numerical answer to the division did not provide the appropriate solution to the problem when the realities of context of the problem were considered. They found that the participants did not always base their responses on realistic considerations of the context situation. Only 28% of their responses contained a realistic solution to the given problem. Contreras and Martínez-Cruz (2003) also examined 139 prospective elementary teachers’ solution processes to additive word problems for which the solution was one more or one less than that produced by simply adding or subtracting the given numbers. They found that about 91% of the prospective teachers’ responses contained incorrect solutions to the problems based on their failure to interpret correctly the solution produced by addition or subtraction of the two numbers given in each word problem. Simon (1990), in his investigation of 33 prospective elementary teachers’ knowledge of division found that they failed to connect their understandings of division and the semantic features of the word problems to the procedures that they employed to divide. Finally, Verschaffel, De Corte, and Borghart (1996) investigated 332 prospective teachers’ conceptions and beliefs about the role of real-world knowledge in arithmetic word problem solving. For each of the 14 word problems, the prospective teachers were first asked to solve the problem themselves, and then to evaluate four different answers given by students. The results revealed a strong overall tendency among the participants to exclude real-world knowledge and realistic considerations from their own spontaneous solutions of school word problems as well as from their appreciations of the students’ answers.

In addition to the above studies that focused on word problems, two studies, Taplin (1996) and Leung (1994), dealt with problem-solving approaches, more generally, and problem-posing processes, respectively. Taplin (1996) explored the approaches to problem solving used by 40 prospective elementary teachers and found that they preferred to work with a narrow range of strategies, predominantly verbal and numerical. They tended to select a method of approach and not change from that through the tutorial, implying inflexibility in their choice or management of problem-solving strategies. Leung (1994) analyzed problem-posing processes (i.e., posing a sequence of problems in each problem-solving activity) of eight prospective elementary teachers with differing levels of mathematics knowledge. Findings showed that those with high mathematics knowledge systematically manipulated given conditions to make problems and used solutions to prior posed problems as new pieces of information to pose subsequent problems.
Those with low mathematics knowledge posed problems that might not be solved mathematically and the mathematics problems posed were not necessarily related in structure. My work (Chapman, 2005) also examined prospective secondary mathematics teachers’ knowledge of problems and problem solving as part of a study to investigate an instructional approach to enhance participants’ knowledge of problem solving. The study indicated that in relation to their initial knowledge, most of the participants made sense of problems in terms of the traditional, routine problems they had experienced, directly or indirectly, prior to entering the teacher education program. They also understood these problems as genuine problems that required thought and logic to arrive at a solution. They understood the problem-solving process in a way consistent with the traditional classroom approach of dealing with routine problems.

These studies imply concerns about how prospective teachers of mathematics may conceptualize problem solving and engage in it. They provide evidence that prospective teachers are likely to need help in their development and understanding of problem solving from the perspectives of a learner and a teacher.

**Instructional Practices for Problem Solving in Teacher Education**

Based on a review of current research in mathematics teacher education for this paper, there seems to be also a scarcity of published studies that explicitly address instructional practices for problem solving in teacher education. In this section, I focus on five studies that include some form of open or non-routine problems that were intended to play a role in the development of prospective teachers’ knowledge of non-routine problem solving and problem-solving pedagogy. Two of these studies are explicitly about problem solving in terms of their stated goals. None defined problem solving, but the implication is that it is associated with a way of thinking involved in solving non-routine problems in the learning of mathematics. These studies provide some basis for instructional practices in helping prospective teachers to grow in their knowledge of problem solving for teaching. This aspect of these studies will be highlighted here in order to make explicit possible ways of engaging prospective teachers in problem solving. In particular, the focus will be on the approaches used in these studies to facilitate learning of problem solving; the goals relating to problem solving of these approaches; and the effect of the approaches on the prospective teachers’ learning of problem solving and its pedagogy. This sample of studies described next deals with approaches to help prospective teachers understand: the nature of problems (Arbaugh & Brown, 2004); the problem solver and problem-solving pedagogy (Lee, 2005); the problem-solving process (Ebby, 2000; Roddick, Becker & Pence, 2000; Szydlik, Szydlik, & Benson, 2003). All of these studies directly or indirectly dealt with problem-solving pedagogy, e.g., how to facilitate students’ learning of problem solving. But there is less focus on teaching through problem solving.

(a) *Nature of Problems (Arbaugh & Brown, 2004)*

Although this study is not explicitly about problem solving, the focus on tasks in the form of problems makes it relevant to problem solving, which begins with the selection of worthwhile problems. The goal of this study was to help prospective teachers understand the relationship between a task and the kind of thinking that task required of students. The tasks consisted of mathematical problems, both routine and non-routine, categorized based on “level of cognitive demand,” i.e., the type of thinking that the task required of students. For example, the category that can be associated with genuine problem solving is “higher-level demands (doing mathematics)” with the following characteristics: require complex and non-algorithmic thinking; demand self-monitoring or self-regulation of one's own cognitive processes; require students to access relevant knowledge and experiences and make appropriate use of them in working through the task; require students to analyze the task and actively examine task constraints that may limit
possible solution strategies and solutions; require considerable cognitive effort and may involve some level of anxiety for the student because of the unpredictable nature of the solution process required (Arbaugh & Brown, 2004, p. 30).

The instructional approach consisted of a task-sorting activity. The researchers developed a set of high school-based mathematical tasks to us in the task-sorting activity in order to help the teachers to learn about the levels of cognitive demands. They used the task-sorting activity with six different groups of prospective high school mathematics teachers.

The effect of the approach was that it was successful in helping the prospective teachers learn the levels of cognitive demand criteria. Each group left their methods class with the ability to categorize tasks according to the criteria, and overall the prospective teachers had been able to communicate the importance of considering the cognitive level demanded of the tasks. They learned to look past surface characteristics of the individual problems and analyze them on the basis of the types of thinking required by students. They built knowledge about the activities that had enabled them to reach a deeper understanding of worthwhile mathematical tasks and the relationship between those tasks and students’ learning.

(b) Problem-solver and Problem-Solving Pedagogy (Lee, 2005)

This is one of the studies that focused explicitly on problem solving with an approach that allowed participants to learn about students as problem solvers and about problem-solving pedagogy. The goal of the approach was to help the prospective teachers interpret and develop in their role of facilitating students’ mathematical problem solving with a technology tool.

The approach consisted of a cycle of “planning–experience–reflection” repeated twice during an undergraduate course to allow the prospective teachers to change their strategies when working with two different groups of students. The prospective teachers enacted the six phases of the cycle by: (1) Individually solving the open-ended problem using a java applet and discussing the problem with peers and the teacher educator/researcher. (2) Developing anticipatory ideas and planning a learning trajectory for students. (3) Interacting with two students as they solved the same problem with a java applet. (4) Discussing the experience with peers, reflecting, and planning a revised learning trajectory for different students. (5) Interacting with two different students as they solved the same problem with a java applet. (6) Reflecting on their role in facilitating students’ problem solving with technology and their understanding of what the students understood about the problem. Several prompts in Phase 2 of the cycle helped the prospective teachers think about students’ learning trajectory by considering possible solution strategies, difficulties students may have, and questions that might be asked to help students overcome those difficulties. In Phase 4, the prospective teachers were asked to reflect on their interactions with students, students’ understanding and problem solving, and changes or improvements desirable for the next group of students. In Phase 6, they were prompted to compare the two experiences and to reflect on what may have caused any similarities or differences in their interactions with the students and how the students solved the problem.

The effects of the approach, based on a study of three prospective teachers, was that the planning–experience–reflection cycle provided opportunities for them to begin to struggle with issues of facilitating students’ problem solving and to make their struggle an open and reflective activity used as an opportunity to improve their practice. Six themes emerged from the cases, i.e., the prospective teachers: (1) Used their own mathematical problem-solving approaches to influence their pedagogical decisions. (2) Desired to ask questions that can guide students in their solution strategies without “giving it all away.” (3) Recognized their own struggle in facilitating students’ problem solving and seem focused on improving aspects of their interactions with students. (4) Assumed the role of an explainer for part of each facilitation phase. (5) Made
pedagogical decisions to use representations in the java applet to promote students’ mathematical thinking or focus their attention on specific aspects of the problem. (6) Used the technology tools in ways consistent with the nature of their interactions and perceived role with students.

(c) Problem-Solving Process (Roddick, Becker & Pence, 2000)

This is another study that explicitly focused on problem solving with an approach that allowed participants to learn about the problem-solving process and some related pedagogical processes. The goal of the approach was to influence prospective teachers' problem solving, problem posing, modelling, and beliefs about the role of problem solving in teaching mathematics. The authors organized two courses aimed at: improving prospective teachers’ problem-solving abilities, their learning of ways to assess problem solving, broadening their views of problem solving and mathematics, and enhancing their understanding of equity issues in teaching mathematics.

In the approaches used in two courses, the prospective secondary teachers were provided with rich and varied problem-solving experiences. They spent significant time on topics such as: what is a problem; examination of problem solving in traditional and innovative curricula; equity issues in problem solving and its assessment; assessment of problem solving and use of technology. Students used a model for reflecting on one’s problem solving (i.e., that of Mason, Burton, & Stacey, 1985) and concentrated on specializing, generalizing, and justifying their work. Both courses included substantial in-class time working in groups on problems and giving presentations and justifications to the class.

The effect of the approach was to impact changes of the participants’ beliefs and practice to various degrees. The participants fell on a continuum ranging from not much discernible implementation to substantial integration of problem solving in their teaching, as in the case of one participant, described in detail. She experienced considerable growth in her views of problem solving and its role in instruction and incorporated such learning into her teaching. The case study demonstrated the changes that can occur in beliefs and instruction as a result of an intensive year-long course that immerses prospective teachers in being reflective problem solvers themselves.

(d) Problem-Solving Process (Szydlik, Szydlik, & Benson, 2003)

This study involves problem solving in an indirect way as part of a mathematics content course for prospective elementary teachers that was designed to provide participants with authentic mathematical experiences and to foster autonomous mathematical behaviors, i.e., behaviors that involve sense-making rather than memorization or appeals to authority. The implied goal of the approach in relation to problem solving was to help the prospective teachers to become autonomous problem solvers by promoting community autonomy rather than autonomy of individuals.

The approach involved engaging the prospective teachers in authentic mathematical behaviors arising out of community work on a set of demanding problems, each of which had an underlying mathematical structure that formed a part of the course content. The problems generally allowed for a variety of problem-solving strategies with a mathematical structure that can be discovered by collecting data, solving a smaller version of the problem, considering several specific cases, or by logical considerations. During a typical class meeting, the prospective teachers worked for 20 to 30 minutes on such a problem in small groups of three or four. The class then convened in a large semicircle for a discussion of their findings, strategies, solutions and arguments. In these discussions, the course instructor emphasized the necessity of mathematical justification; complete solutions required logical arguments. The class was designated as the mathematical authority. The instructor declined to give the final word on the correctness or completeness of any solution and there was no text. The instructor also provided almost no assistance in the problem solving aspect of the course and no answers were provided for problems. The only way for the class to
understand a problem was to figure it out. The only way to know they were correct was to find a convincing argument. However the spirit of the class was consistently one of community inquiry. On six occasions throughout the semester, the prospective teachers were asked to produce written reports that focused on their mathematical thinking by describing the problem, discussing the strategies they used to work on the problem (including those that did and did not lead to a solution), providing a solution, and, finally, arguing that their solution was complete and valid. In some cases these reports were produced as group projects. These reports provided opportunities for further reflection and discussion.

The effect of the approach was that a classroom focusing on problem solving using a variety of strategies, reflection on the process of problem solving, and engagement in the process of exploration, conjecture, and argument can help prospective teacher develop mathematical beliefs that are consistent with autonomous behavior. The community work on the problems made the process less frustrating for the prospective teachers, allowed them to see the ways in which their peers did mathematics, and showed them that problems could be solved in more than one way, i.e., a broadening in the acceptable methods of solving problems. The participants’ beliefs became more supportive of autonomous behaviors during the course.

(e) Problem-Solving Process (Ebby, 2000)
This study involves problem solving in an indirect way as part of a methods course in a teacher education program that aimed to integrate fieldwork and coursework. One goal of the approach studied that relates to problem solving is helping prospective teachers to draw on their experiences as learners of problem solving in developing a conception of their teaching role. Thus the approach provided opportunities for them to learn about problem-solving and mathematics pedagogy.

The approach involved inviting the prospective teachers to participate in mathematical inquiry through individual, cooperative, and whole-class problem solving. Each week the professor assigned a non-routine problem-of-the-week and encouraged the prospective teachers to work together on the solutions outside of class. In class, the professor asked for volunteers to share their solutions and strategies with the rest of the class. These problem-solving activities were designed to invite prospective teachers to learn mathematics in a non-threatening environment and to reconsider what it means to learn, teach, and know mathematics. By focusing on solution process over answer, the professor was also modeling an alternative pedagogy for them.

The effect of the approach based on the three participants studied was to provide them with a different experience. One participant experienced what it was like to be an active agent in her own mathematical learning through her engagement in problem solving. Another discovered that others had a diversity of approaches to mathematical problems and that their understandings were often different from hers. Another developed a new definition of the nature of mathematics as a result of her engagement in mathematical problem solving in the methods course.

The Author’s Multi-goal Approach
This section deals with a study I have conducted that explicitly deals with problem solving as its primary focus. It is an extension of the instructional approach reported in Chapman, 2005. It incorporated what was learned then to further enhance the approach. Thus the goal of this follow-up study was to investigate the extended approach to help prospective teachers to understand problem solving as a mathematical and pedagogical process, i.e., as mathematical thinking and a method of instruction, respectively. This is a multi-goal approach because it explicitly deals with: development of understanding of problems, problem solver, problem-solving process, problem-solving pedagogy and problem solving as inquiry-based teaching. What follows is an abbreviated report on the study to highlight the nature and effect of the instructional approach.
Theoretical Perspective of Approach: The approach is framed in the work of Dewey (1933) with a focus on inquiry, reflection and social interactions; cognitive guided instructions (Carpenter et al., 1999) with a focus on understanding students’ thinking and strategies to inform instruction; and narrative inquiry (Polkinghorne, 1988), with a focus of using personal experiences to understand self. An inquiry perspective framed in social constructivism formed a basis of the learning activities used in this study.

Description of the Approach: The activities were organized in three stages: individual reflection, inquiry activities and final reflections.

Individual-reflection: The first stage focused on individual self-reflection on problems and problem solving in order to create awareness of each of the prospective teacher’s initial conceptions and knowledge. The participants were required to respond to a list of questions and prompts in sequence that included: What is a problem? Choose a grade and make a mathematics problem that would be a problem for those students. What did you think of to make the problem? Why is it a problem? Is it a ‘good’ math problem? Why? What process do you go through when you solve a problem? If possible, represent the process with a flowchart.

Inquiry activities: The second stage consisted of inquiry activities intended to extend the prospective teachers’ initial conceptions and knowledge. The prospective teachers worked on all problems in these activities without the facilitator’s intervention. These activities included:

1) Comparing different types of problems without solving them in order to explore the nature of problems used in teaching mathematics and the goal of these problems in learning mathematics. The prospective teachers were provided with a list of different categories of problems influenced by Charles and Lester (1982). They were asked to compare and contrast the problems and to draw conclusions about problems in learning/teaching mathematics.

2) Writing and unpacking narratives of their experiences in order to examine the cognitive and affective components of the behaviors involved in solving a non-routine problem. The prospective teachers were required to write narratives of their experiences solving a problem that was assigned to them. The narrative had to be a temporal account not only of the mental and physical activities they engaged in to solve the problem, but the emotional aspects of the experience. They later analyzed the narrative in terms of the affective aspect of the experience.

3) Investigating others (e.g., peers and secondary school students) solving non-routine problems to explore the thinking of others compared to their own. The prospective teachers were required to solve an assigned problem and make notes of the thought process. They then worked in pairs, took turns to observe each other solve the problem while thinking aloud, and compared their thought processes. They selected a non-routine problem appropriate for a secondary school student, first solved it, and then used it to observe the student solving it while thinking aloud. They also solved a problem as a group in order to explore the collaborative and cooperative problem-solving experience.

4) Developing a model for non-routine problem solving, representing it as a flowchart, and applying it to solving a non-routine problem in order to evaluate it.

Final reflection: The third stage included activities that required the prospective teachers to engage in a final reflection by comparing their post-Stage 2 thinking with their pre-Stage 2 thinking; comparing their understanding of problems to theory (e.g., Charles & Lester, 1982; NCTM, 1989); comparing their problem-solving models and flowcharts with those from theory provided to them (e.g., Mason, Burton, & Stacey, 1982; Polya, 1954; Verschaffel, Greer, & de Corte, 2000); relating their problem-solving models to an inquiry instructional model for teaching secondary school mathematics; applying their knowledge to critique a current secondary school mathematics textbook approved for use in the Province; and preparing a lesson plan based on their inquiry instructional model.
Group reflection: Each of the three stages also required small group and whole-class interactions. This included: participants sharing and comparing their individual reflections in Stages 1 and 3 and their findings from the inquiry activities in Stage 2; preparing a model of problem-solving or inquiry-based teaching in small groups; and sharing and discussing small-groups’ findings in a whole-class setting.

Research Method: The participants were 29 preservice secondary mathematics teachers in the second semester of their 2-year post-degree education program. This was their first course in mathematics education, so they had no instruction or theory on problem solving prior to this experience. They also were not taking any other mathematics education course in this semester. The reflection and inquiry activities served both research and learning purposes. Thus data consisted of copies of all of the participants’ written work for all of the activities. There were also field notes of their group discussions and whole-class discussions. The analysis began with open-ended coding (Strauss & Corbin, 1998) of the data. The researcher and research assistants, working independently, coded the data. Coding included identifying significant statements about the participants’ thinking of problems and problem solving and the changes in thinking resulting from the activities. The coded information was categorized based on common themes and frequency of occurrence to form the findings.

Findings of Effects of Approach: The approach was effective in expanding and deepening the participants’ understanding of problems, problem solving, problem-solving pedagogy and inquiry-based teaching. Their thinking of problems shifted from predominantly traditional exercises or word problems to an understanding of characteristics that constitute worthwhile mathematics problems. Some of their descriptors of good problems are: they are of many different forms and types, challenging, needs one to use deductive or inductive reasoning to come to a solution, have one or more solutions and many approaches, interesting, not procedural and memorization of facts, challenging but solvable for grade level, applicable to real-life problem, often requires more than one attempt to find a solution.

The participants’ description of the problem-solving process was also enhanced, as reflected in their flowcharts of it, which showed the need to move back and forth, as opposed to their initial thinking of a linear path, to get to a solution. Following is a participant’s problem-solving model.

Any Assumptions that need to be made
Do the assumptions remove a barrier?
Try to do with and without barrier

How does it fit with assumptions made?

Understand the problem
Reread, write in own words
List what you know/is given
What is being sought as solution?

Plan – establish relationships, patterns, other representations
Follow through to see if it yields an answer
Rethink/replan if doesn’t answer the question

Check solution/Reason it out
Does it make sense? Is it an answer?
Have you learned anything from the plan/solution?
Can you apply what you learned to other problems?
The theoretical problem-solving models the participants examined in Stage 3 of the approach allowed them to validate, and in some cases refine parts of, their models, but not to change it, showing preference for their experience as a basis of validating what was more meaningful to them. Most of the participants viewed Polya’s model to be closer to theirs and easier to follow and use in their teaching. One explained, “It flows more smoothly, it is cyclic, and non-exclusive in its processes. Although very generic in describing the steps, it reflects my cognitive steps in problem solving. As well, the steps I take in developing a plan varies from problem to problem.” In general, the inquiry activities of Stage 2 of the approach allowed the participants to construct knowledge compatible with formal theory of problem solving. This allowed them to relate to the theoretical approaches they examined in Stage 3 of the approach in a more meaningful way.

In terms of problem-solving pedagogy, the thinking of the prospective teachers shifted to a student-centered approach. For example, they explained that they will have students work on a problem first, then share and discuss it in a whole-class discussion, or will have a whole-class discussion first to understand the problem, have students work in groups or individually, then share and discuss. They will pose questions or prompt the students when they (students) are stuck or in response to their (teachers) questions. They would use their problem-solving model to help to frame prompts and guide the whole-class discussions. This approach they viewed as being similar to inquiry-based teaching where they will start with an applied situation that embodied the mathematics concept being taught and allow students to unpack it with their (the teacher) guidance to understand the concept.

Overall, the study suggests that this three-stage approach framed in a social constructivist perspective and providing inquiry-based and self-reflective opportunities involving non-routine problem solving, followed by comparison to theory can help to deepen prospective secondary teachers’ understanding and knowledge of problem solving in their teaching. However, whether they are able to enact this knowledge in their teaching has not been researched as yet. But holding such knowledge seems like an important first step in getting there.

Implications for Teacher Education

Based on the literature review and my study in the preceding sections of this paper, following are eight key characteristics of instructional practices identified as important to form a basis of preservice teachers’ learning about problem solving. (1) Exploring others as problem solver: e.g., the prospective teacher works with a child or peer to observe, interview, and document information about this child or peer as a problem solver. (2) Exploring self as problem solver: i.e., inquiring into one’s thinking, learning and instructional practices and developing ability to monitor and to control one’s activities when solving problems. (3) Exploring nature/structure of problems. (4) Solving challenging problems individually and in small groups without external assistance, e.g., to develop awareness of strategies and skills for solving problems. (5) Posing problems. (6) Comparing self with others, e.g., peers, students, theorists. (7) Formulating instructional model for problem solving. (8) Exploring self as facilitator of problem solving, i.e., to develop an understanding of the teacher’s role in facilitating students’ problem solving.

If teachers are to hold knowledge to help them to teach about and through problem solving, they should be provided with experiences not only in solving problems, but also with all of these eight characteristics. It may also be necessary for these characteristics to be embodied in an integrated experience as opposed to being in isolation.

References
