Digital Technologies and Problem-Solving Practices[1]

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Introduction

During a trip in the subway, in Mexico City, one can observe how people become isolated whilst using their phones for messaging and playing games. Isolation, however, is apparent as the phones *mediate* their presence in another place. Identity involves presence in social space, but now, social space is extended, on a permanent basis, into a realm of virtual reality. The phone is the key to enter and participate in this enlarged infrastructure of society.

Friedman (2005) uses the notion of *flatness* to explain and argue that with the use of technology resources, more people are involved or can directly collaborate in addressing and discussing society concerns than ever before. He also recognizes that in less than a decade, since the publication of his book, the new digital technologies have transformed and are transforming human relations and human cognitive powers.

At the first moment, the use of cognitive technologies (Pea, 1985) could be seen as *amplifiers* of human cognition. For instance, the use of a handheld calculator with Computer Algebra Systems (CAS) can help us solve problems that involve finding the roots of a given polynomial. This is something we could do without the handheld device but it is faster and convenient to rely on this recourse. Like a magnifying glass, a cognitive technology can improve an ability we already possess. People usually develop this cognitive affordance when they begin representing and exploring tasks through this technology. However, in the long run, this is not quite the only proper role. Like a Trojan horse, a cognitive technology begins working stealthily in our mind and after a while it becomes part of our cognitive resources. This is the case of the technology of writing, for instance. As M. Donald (2001, p. 302) has explained, literacy skills transform the functional architecture of the brain and have a profound impact on *how literate people perform their cognitive work*. The complex neural components of a literate vocabulary, Donald explains, have to be hammered by years of schooling to rewire the functional organization of our thinking. Similarly, the decimal system (Kaput & Schorr, 2008, p. 212)

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first enlarged access to computation and eventually paved the way to Modern Age. Today, we cannot imagine the world without these technologies. They have become infrastructural —obviously much more than mere amplifiers. That is, they have become essential tools that everyone learns and uses to sustain individual and social activities. They have rewired, as Donald wrote, the functional organization of individual brains and, at the same time, have become coextensive with our culture.
It is the omnipresence of technologies in society that eventually endow them with *invisibility*: they blend into society, as people are increasingly accustomed to their effects.

We need to develop the critical capabilities to translate scientific and technological developments into our realities, more importantly, into our *educational* realities. Scientific knowledge undergoes ostensible transformations before entering into the classrooms. One comprehensive framework to guide and understand this translation of scientific knowledge into the knowledge taught in schools is provided by the theory of *Didactic Transposition* (Chevallard, 1985). Put simply, didactic transposition includes ways to reorganize that knowledge so that the new resulting version is available as educational material.
This transposition creates a tension between social expectations, on one side, and what an educational system can deliver and offer to learners. It is important to recognize that a school culture always leaves significant marks on students and teachers' values. As Michelle Artigue (2002, p. 245) has aptly expressed, "these [culture] values were established, through history, in environments poor in technology, and they have only slowly come to terms with the evolution of mathematical practice linked to technological evolution". However, there is a fact that must be singled out: The emergent knowledge produced through the digital media is different from the knowledge emerging from a paper-and-pencil medium because the mediating artifact is not epistemologically neutral. That is, the nature of the knowledge is inextricably linked to the mediating artifact (Moreno-Armella & Hegedus, 2009, p. 501). We will have an opportunity to discuss this issue broadly, later in this chapter.

It is important to recognize the existence of a natural tension between the past and the future, but it is also possible to resolve it if we realize that the prudent face looks into the past, and the innovative face looks into the future. Today, our students are increasingly digital natives and as teachers, we are digital immigrants (Prensky, 2010). Yet, even if we speak of (digital) technology with an accent, we need to blend past technologies with the new ones.

In this context, school culture requires a gradual but permanent re-orientation of its practices, and of its cognitive and epistemological assumptions, for students to gain access to powerful mathematical ideas. In our view, the classroom should be conceived of as the central nervous system of the educational process. However, that classrooom, as well as the educational system *in toto*, are open systems and consequently are under the multidimensional influence of its social and cultural environments.

Today, we have new ways to represent and communicate our experiences, in particular, to communicate the knowledge we have acquired. For example, communication technologies facilitate not only direct interaction among research communities; but also the sharing of experiences and results. Bottino, Artigue and Noss (2009) present a collaboration project that involves several European research teams discussing goals and ways to frame technology-enhanced learning from different theoretical traditions. But all this is not just about knowledge; it is centrally about *knowing*. As Schmidt and Cohen explain in their new book (2013) a computer, in 2025, will be sixty-four times faster than it is in 2013. This is a huge increase of cognitive technologies that help individuals reorganize their ways of thinking including their problem solving approaches. We cannot foresee, today, what this would imply for society in general and for education in particular in the next decades. Mathematics is part of our culture and lives; it is embedded in every digital artifact, phones, computers, eBooks, and so on. Eventually, we are compelled to ask: What is the *new* role of mathematics in contemporary societies increasingly saturated by the use of digital artifacts? How can we use available technologies (including smart phones) to foster students development of sense making actitivies and reasoning? Thus, we are forced to understand the strategies that teachers follow to appropriate the digital artifacts at their reach. For instance they can use *conveyance technologies* or *mathematical action technologies* (Dick & Hollebrands, 2011). The former allow the teacher to present or communicate mathematical ideas in the classroom. Even if these technologies are not mathematics specific (PowerPoint, LCD projectors, for instance) they are important for integrating the classroom around the discussion of someone's point of view with respect to a mathematical idea.

Mathematical action technologies (Dick and Hollebrands, 2011), on the other hand, are used to activate and improve exploration, conjecture formulation, argumentation, and in general, mathematical ways of thinking.

A focus on mathematical tasks

In the last 10 years we have consistently been involved in several national research projects that aim to analyze and discuss the extent to which the use of digital technologies provide teachers and learners with new avenues to grasp and develop mathematical knowledge (Moreno-Armella & Santos-Trigo, 2008). During the development of those projects we have addressed themes related to teachers involvement in problem solving activities that enhance the use of several digital tools, curriculum reforms, and ways to design and implement mathematical tasks in actual learning scenarios (Santos-Trigo & Camacho-Machín, 2009). Our research approach includes working directly with teachers at public institutions through seminars and workshops. There, they have an opportunity to identify and discuss international developments around the use of digital technologies such as those published in handbooks and research journals and ways to frame them in their actual teaching practices. Indeed, several of the tasks used in this chapter to illustrate ways of reasoning that emerge when learners think of and approach the tasks through the tools' affordances came from those projects. That is, tasks play an important role not only in fostering learners' construction of mathematical knowledge; but also in documenting students' ways of reasoning associated with the use of digital technology.

Guide and being-guided by an artifact

A musician develops her creative skills through a profound, intimate involvement with her musical instrument. When the musician is a professional, there is no border between this artist and her instrument. This is what one feels whilst listening to Jacqueline du Pré's rendering of Elgar's cello concerto. Nothing of this would be possible without the mutual dovetailing of Du Pré and her cello. We mean, the conceptual image of the cello that Du Pré was able to *internalize* along years of hard reflective practice. There is fluidity in this human- artifact integration that transforms her artistic skills making the cello acquire a characteristic sound. There is a somewhat similar process that takes place when we learn to read. At the earlier stages our reading is awkward, slow, but with time a feeling of satisfaction awakens as our reading skills grow and we can read smoothly a whole text. A deep and complex process takes place in our cognitive structures as this process develops. M. Donald (2001, p. 302) has explained that literacy skills transform the functional architecture of the brain and have a profound impact on *how literate people perform their cognitive work*. The complex neural components of a literate vocabulary, Donald explains, have to be hammered by years of schooling to rewire the functional organization of our thinking. Similar process stakes place when we appropriate numbers at school. It is easy to multiply 7 by 8 but if we want to multiply 12345 by 78654 then we write the numbers and follow some specific rules. It is because we have been able to internalize reading and writing and the decimal system, that we are able to perform the corresponding operations. It is as if these skills have always being there as components of our cognitive system. Indeed, we know that our cognitive work is mediated by the alphabet, by the decimal system and by each one of the cognitive technologies we internalize through our lives. Collecting these examples help us find the meaning for the expression *mind is mediated mind*. It is the omnipresence of technologies in society that eventually turns them *invisible*: people become familiar with their presence that, in a certain sense, they begin considering those technologies as an inherent part of the social landscape. Take electricity, roads and the phone infrastructure as examples. Technologies (not simply *devices*) *reformat* societies. Educational systems have been under the influence of a great deal of technologies, for instance, the ruler, the compass, the pencil, the table of logarithms. Even the slate can be considered a technology whose influence we can appreciate in the reorganization of the classroom space. Today the digital technologies are present in this classroom embodied, for instance, in tablets, computers, and films. However, they do not provide by themselves the answers we look for the complex problems pertaining to education.

This short narrative aims to describe the nature of the relationship between a society or a person and an artifact that they/she want(s) to use for accomplishing a task. There is a deep level of complexity: technical, cultural and cognitive implicit in this narrative. We shall try to reveal a significant piece of this complexity in the following pages when we reflect on learning and teaching mathematics with the mediation of digital technologies.

Working with teachers in our graduate program has been an invaluable opportunity to learn how they deal with digital technologies like Geogebra, installed in computers, for instance. They are motivated because they will use this technology when they return to their classrooms. There is a professional commitment as well as an increasing social pressure to gain fluency with these artifacts — we find cell phones in the subway as well as in the schools. In the first decades of the present century there have been serious efforts to *problematize* the presence of CAS and Dynamic Geometry Software (DGS) in the classroom. Besides been installed in material artifacts (calculators, computers, smartphones, and so on), CAS and DGS are semiotic artifacts because they *mediate* semiotic tasks when we are, for instance, trying to coordinate several symbolic registers of a mathematical object. We can illustrate this with the analysis of area variation of a family of rectangles with fixed perimeter through a discrete approach (a table); a graphic generated via loci of points, or in terms of an algebraic model.

Teachers need to understand the workings of these artifacts, and their syntactical rules, in order to use them meaningfully as *mediators* of mathematical knowledge. For this to happen, there must be a melody to be played, that is, teachers need a mathematical task. The task is an incentive for the teachers to figure out how to integrate in meaningful ways the symbolic artifact to their own intellectual resources in order to solve that task. If a person succeeds in integrating the artifact to her cognitive resources to solve a task, then, Verillon and Rabardel (1995) explain that the artifact has become an *instrument*. For instance, when we compute the multiplication of two large numbers our cognitive activity is mediated by the positional system we use to represent numbers. We find it very natural to proceed as we usually do. The positional system in base ten is more than a cultural artifact: it has become an instrument of our reason to deal with numbers. This is in sheer contrast to computing with numbers written in base, say, seven. In this case we have a cultural artifact that most people have not transformed into an instrument to think in numbers and solve problems.

Vignettes and Exemplars: Geometry and Calculus.

We cannot forget that today's curriculum has deep roots in the ways mathematics has been conceived traditionally with paper and pencil. And we cannot forget either, the importance of the available digital armamentarium, resources that teachers can incorporate into their teaching practices. But even if digital technologies are not fully integrated within the school mathematical universe, they will gradually erode and transform the mathematical ways of thinking embedded in the traditional system. Like the two faces of *Janus* that looks to the past and to the future, Education looks to tradition and innovation.

To consider this dilemma, tradition/innovation, we explore digital representations of mathematical entities. Doing so will reveal properties of these entities that lie hidden or opaque, to begin with. Our goal is to develop a *transitional way of thinking* more in agreement with the requirements of education today. For instance, simple mathematics objects such as the perpendicular bisector that appears in elementary education are re- conceptualized when they are explored dynamically (see the task in the following section). Indeed, this concept becomes crucial to generate and explore, for instance, conic sections.

We cannot conceive of transforming education without the conscientious efforts of the teachers.

Let us begin now presenting some exemplars pertaining to *mathematical ways of thinking* for teachers.

A re-conceptualization of the perpendicular bisector.

For our work, we decided that *understanding* meant the moment when a cultural artifact (Trouche, 2004) (as the perpendicular bisector) became an instrument integrated with other cognitive instruments. Yet, this was rather restricted, so we searched for the moment the teachers in the classroom were *aware* of the instrument and they could use it to solve a task. We offered the following task for exploration:

Let us consider the segment AB and its perpendicular bisector as shown below.

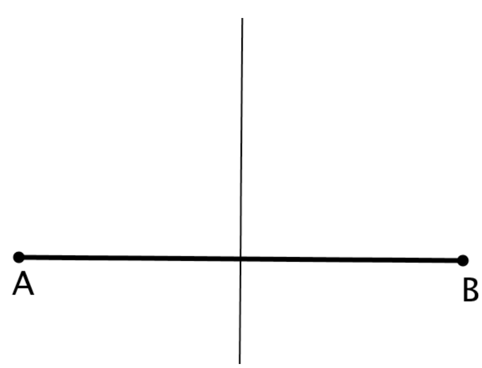

Figure 1: Perpendicular bisector

At this point the idea was to use the perpendicular bisector as an *organizing principle* to explore conic sections. We suggested the construction of triangles with the third vertex C on the perpendicular bisector and then asked them to explore the locus of D, (D is the intersection point of the perpendicular bisector of side BC and the bisector of angle CAB) as C travels along the perpendicular bisector of AB (see figure 2). Teachers got a locus that looked like a conic section. Now the problem began: Is it a conic section?

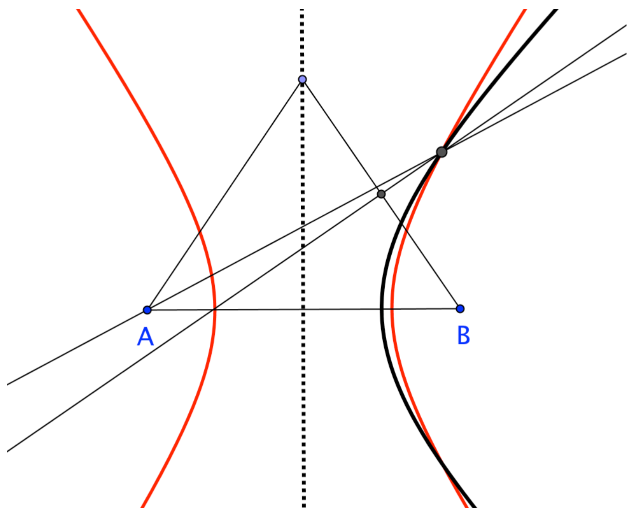

Figure 2: "Conic" section

The figure does not do justice to what happened next. At that moment teachers were puzzled: The question was unexpected. They had worked with conic sections using the traditional analytic expressions and now, where were the coordinate axes? After a while a teacher, Victor realized they could try "to cover" the locus with a conic section passing through five points of the locus. The Geogebra dynamic environment provides a command to draw a conic that passes through five points and teachers had used it extensively. Following this idea, teachers understood (after a mediating discussion) that the way to disprove the conjecture was *legitimate,* as they had used something *infrastructural:* the conic through five points. They found the conic that disproved the result by dragging and rotating the vertices and the segments in the figure. This is crucial: they dragged, rotated the figure *preserving the underlying structure*.

We thought it was more productive to begin with a problem that would find a solution by means of a counterexample. The lesson learned was: dragging is a mediator for exploring, and as a teacher, Laura, said, "if something is true you cannot *destroy it* by dragging it." The window to structure was opened. Next session, Alvaro came with this construction:

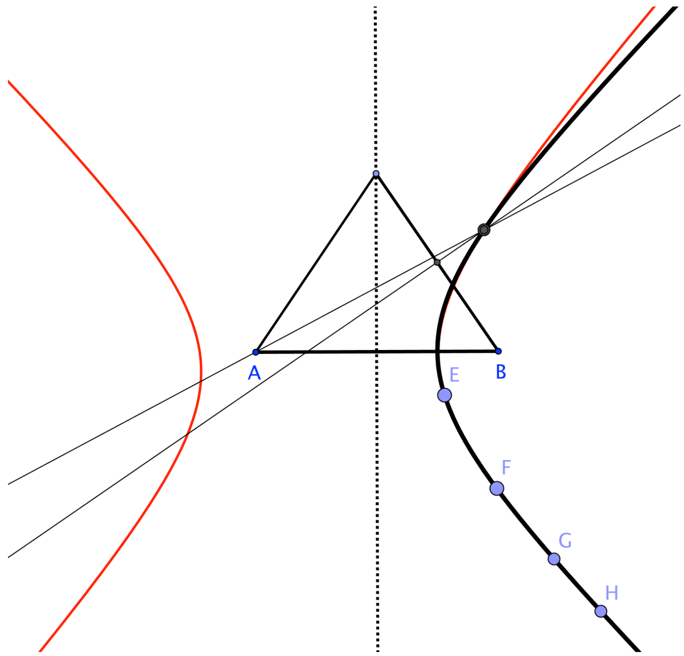

Figure 3: Another counterexample

His five points for the conic were E, F, G, H, I, then he began playing with the construction by moving point I along the locus and observing the different conics.
The way to introduce the five points construction as an infrastructural artifact in the digital medium was establishing the similarity with: (i) two points determine one straight line; (iii) three points determine one circle. Then we discussed the fact that mathematics was embedded in the medium.

Discovering the parabola. Teachers were very enthusiastic with the experience of solving a problem by means of a counterexample. Next time, we decided to try a variant of the former exemplar. Instead of taking the bisector of angle CAB, we suggested to work with the intersection point of the perpendicular bisector of CB and the perpendicular to line b at C as shown in the next graphic:

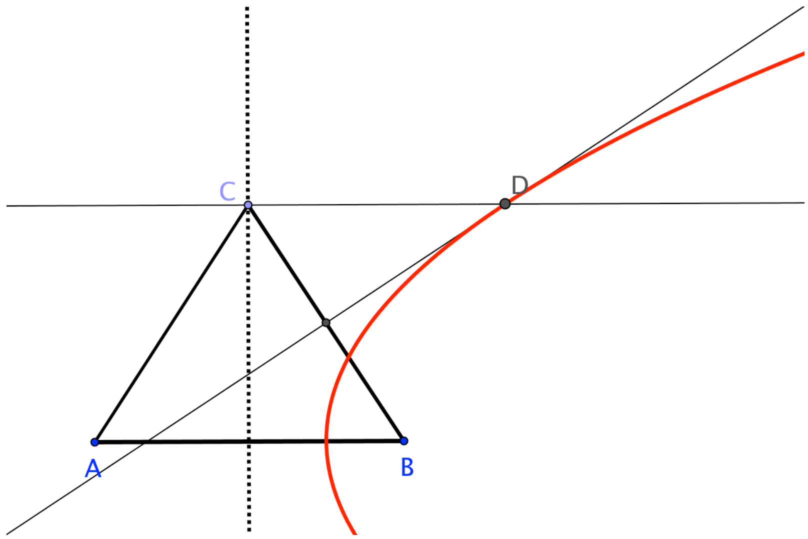

Figure 4: locus of D as C travels on b

This time the locus is a parabola and the straight line b is the directrix, and the focus is point B. Teachers had already worked with the definition of the parabola as the locus of points equidistant from a line (the directrix) and a point (the focus). They identified *b* as the directrix and B as the focus, but what was unexpected to us, was the fact that, to identify the locus as a parabola, they rotated the figure a right angle to the left:

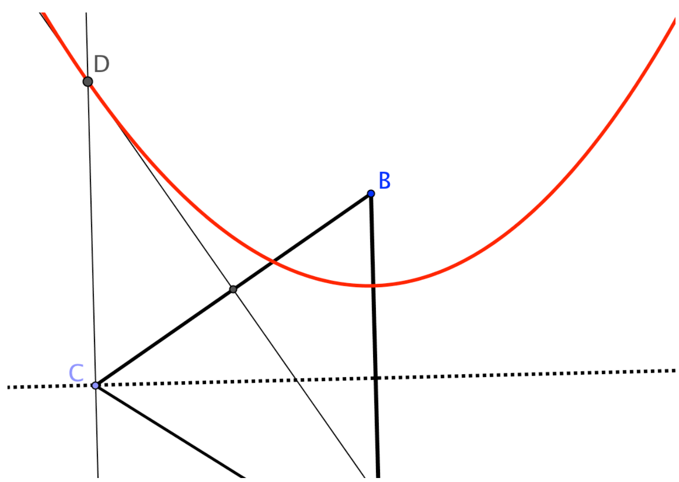

Figure 5: Identifying the locus

They hid the segments and points that were not relevant for the original definition of the parabola as a locus. We were wondering why they had to rotate the graph as it was clear that the locus as shown in figure 4 is a parabola. It was clear for us but *not* for them: The definition of parabola *always* comes with this graph (figure 5), so an inertia is created due to the fact that the graphical representation of the conic reflect *our own body*, as when we draw on the slate. There is a sense of vertical and a sense of horizontal that are present when one tries to recognize a shape. It is clear that this event illustrates the *embodiment of knowledge* (Moreno-Armella and Hegedus, 2009).

Thus, simple mathematical entities such as triangles or circles can be represented digitally and become a platform or departure point to identify and explore more complex entities. This is the case with the following exemplar:

Looking for the hidden conics. Draw a circle with center C. On the circle we choose point E and draw line CE. Then, we select point F on line CE and draw the segment FG. We take the perpendicular bisector of FG. This perpendicular bisector intersects line CE at H (Figure 1).

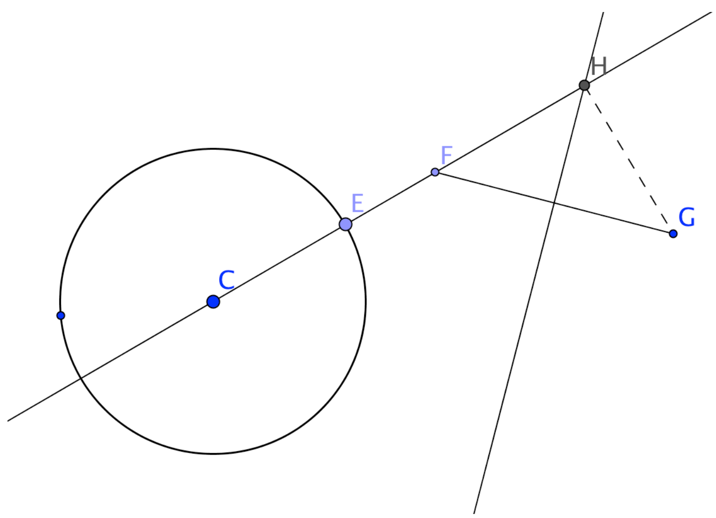

Figure 6: Dynamic triangle and perpendicular bisector

We asked: What is the locus of point H when point E travels the circle?
Figure 7 shows that the locus seems to be a hyperbola.

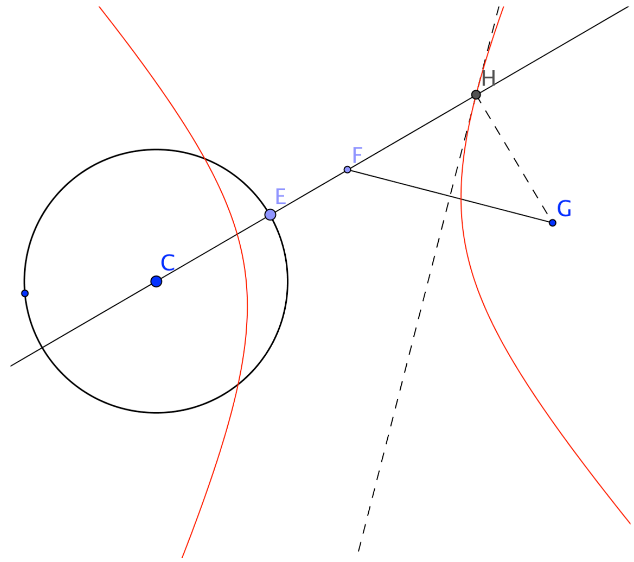

Figure 7: What is the locus of point H when point E travels along the circle?

Drawing the locus of H is an infrastructural affordance of the environment. In this context, it is a point-and-click action. Teachers were amazed with the environment's answer. Indeed, the locus *seemed* a hyperbola. But, was it? At this time, teachers were almost lost; they could not devise a plan of action. At our suggestions they measured the distances involved and found that the segment HF was congruent to segment HG. But —and this was Manuel's conclusion— that always happens because H is on the perpendicular bisector of FG! They drew the segment HG and things became clearer. It took another half an hour to write:

It is observed that (for every position of H):
d(C,H) — d(H, G) = d(C, F)
Because d(H, F)= d(H, G). Consequently, the locus is a hyperbola.

We want to emphasize that the possibility to drag points and observe the smooth morphing of the locus was instrumental for reaching the right conclusion. In this case the moving point was F. That made the teachers to propose that C and G as the foci of the hyperbola. There is no doubt: *Motion is worth a thousand pictures*.

It was visible that moving point G (this time F was fixed) different loci were obtained. We observed that they were moving G afar from the circle so we decided to ask: *What happens if G gets closer to the circle?*

We kept quiet for a while as they discovered that *suddenly* the hyperbola morphed into a figure that seemed an ellipse.
Teachers found astounding this sudden morphing into an “ellipse” when point G got closer to the circle. They had *proved* after a while playing with the resources provided by the environment that d(C, H) — d(H, G) was a constant equal to d(C, F). Now, when the morphed figure seemed to be an ellipse

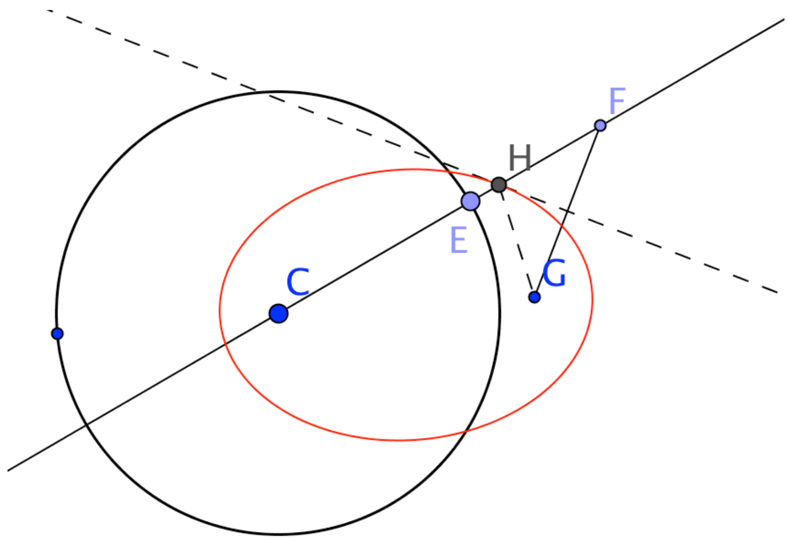

Figure 8: Ellipse with foci C, G

It was not the difference but the sum: d(C, H)+ d(H, G)= d(C, E), a constant for every position of H on the perpendicular bisector of segment FG. That made clear for them that the locus was, indeed, an ellipse. Eventually, teachers came to perceive that the position of F on line CE has “the key” (their words) to decide if the conic was a hyperbola or an ellipse.
We thought, at that point, that it was timely to simplify the construction by identifying points E and F. Then we asked the teachers to figure out how to draw a tangent to the ellipse from any point C inside this circle (see figure 9).

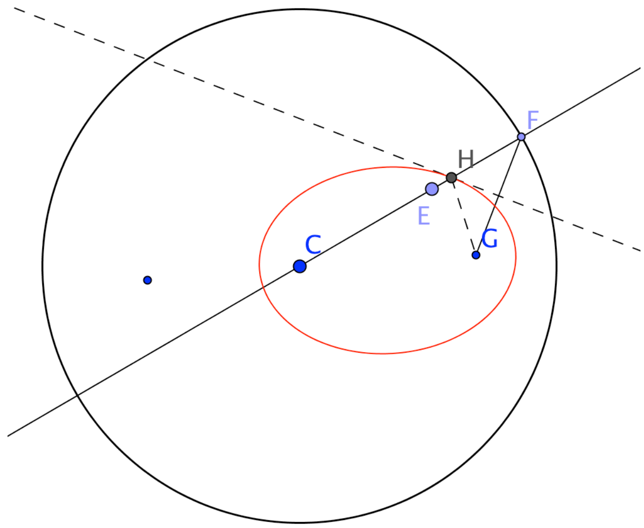

Figure 9: The new construction

We will omit this discussion, which completed a basic *dynamic* analytic geometry course, as we want to share a couple of Calculus exemplars we discussed with other group. However, we consider important to offer some reflections based on the previous teaching-teachers experience before the Calculus exemplars.

A brief reflection. The point F (see Figures 8 and 9) is a *hot-point* (Moreno- Armella and Hegedus, 2009) because if we keep fixed all other points, in these constructions, point F controls the underlying structure of conics we can display. What is really central is that the environment provides these points in every construction. Emphasizing this idea made teachers to become aware that what we have on the screen are not simply dynamic drawings but geometric *structures*. It was the movement that made visible the structure. The structure is hidden behind any particular rendition on the screen. One of the teachers, Laura, mentioned that the idea that the structure is visible through movement was similar to camouflaged objects: If a moth is standing still on a tree, then the bird (its predator) cannot see the moth, unless the moth moves. Of course the similarity ends here as *seeing the structure* is a very complex cognitive process: What one sees, through the digital, executable representation is a conceptual entity (Moreno-Armella and Hegedus, 2009). At this point we pondered over the pertinence of going back to the discussion on the nature of mathematical entities. We found that working in a dynamic environment where learners could drag a figure, and modify the length of a segment, for instance, gave them an opportunity to explore the behavior of a family of objects within the same configuration.

In all the preceding exemplars, the basic geometric construction has been the perpendicular bisector. This construction is the *organizing principle* to explore conic sections the way we chose to follow.

Action does not belong (exclusively) to the user and neither does it to the environment; both, user and environment are actors and re-actors. We understand dragging as our hands within the environment, where it is possible to *touch* and transform mathematical entities living in the digital environment. User and environment are, from the point of view of agency, *coextensive*. Thus, we speak of *co-action* between the user and the environment, not just between the user and the artifact. Co-action is the broader process within which an artifact is being internalized as a cognitive instrument. Yet, in the social space of the classroom there can be a collective actor. One participant can observe how another drives the technology at hands and, the former, incorporates into her strategies what she observed. At the end participants can act and react to the environment in ways that are essentially different from their initial ones. We can learn *from*, *through* and *with* the others. So the traditional triangle user-technology-task has to be enlarged: Co-action becomes embedded in a social structure: Culture cannot be factorized from the technology appropriation processes, and technology cannot be factorized from culture.

Two exemplars from Calculus

Digital media, with their *executable* representations (Moreno-Armella, Hegedus, and Kaput, 2013, p. 105), have transformed the traditional mathematics of change and accumulation. There is a profound cognitive difference between applying a geometric transformation, *on paper*, to rotate or dilate a triangle, where all the action takes place in human imagination and applying that transformation through its executable version and perceiving it on the screen. Thus, variation, change, and accumulation are not anymore restricted to the written version of Calculus. However, paper-and-pencil tradition cannot be ignored and left aside. We have to allow its representational re-description in terms of digital representations. In fact, there are many mathematical entities that can be re-described, translated into digital environments and explored there.

Area approximation. Pierre de Fermat (1601-1665) solved in a very original way, the problem of computing the area under a parabola $y=x^p$, p=/1. Fermat began by subdividing the interval [0, 1] into an infinite sequence of subintervals with end-points of the form z^n, with n=1,2,3,... and Z a fixed number 0<z<1. That is, Fermat used an infinite subdivision of the interval by means of a geometric progression.

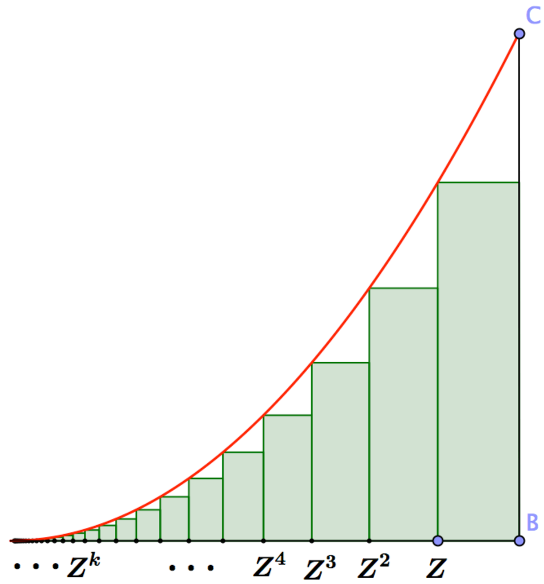

Figure 10: Infinite subdivision

It takes some work to find a closed expression for the sum of the areas of the rectangles; Fermat did it. Afterwards, Fermat's reasoning was to eliminate the error, that is, to fill the white triangles over the rectangles. At this point, we decided to stop the narrative (indeed, avoid the computations) and ask the teachers, taking into account their already gained experience with digital environments, how could they *explain* Fermat's result. They had some experience working with sliders; the answer came after a collective and very emotional discussion (Vizgin, 2001) in the classroom. Let us see the next figure:

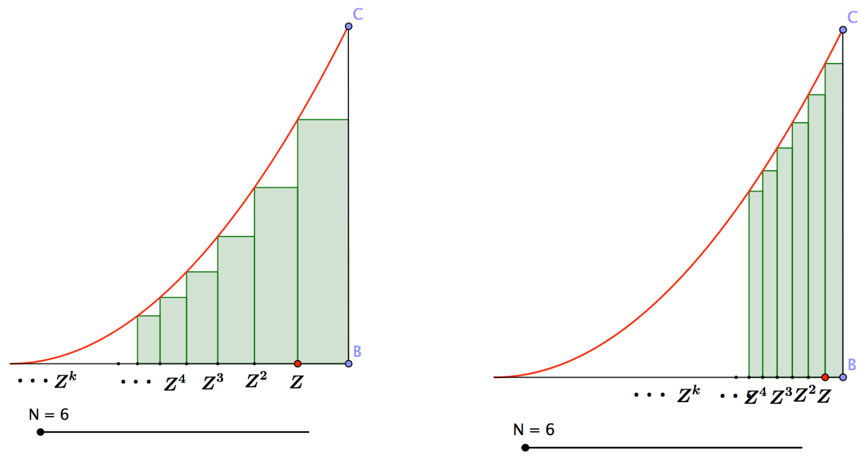

Figure 11(a)

The figure on the left shows a fixed value of Z<1, and six rectangles. The slider is used to control the number of rectangles. The figure on the right shows six rectangles but now, the value of Z is closer to 1. One can observe that the process of approximation depends not only of the number of rectangles (*in principle*, there is an infinite number of rectangles) but of how close to 1 is Z. If Z is closer to one then one needs more rectangles.

We certainly believe that the teachers were able to understand the basic idea behind an approximation process notwithstanding the contextual constraints. We have called this kind of particular contexts, *domains of abstraction*: There is something general “hidden” below the context in question (Moreno-Armella and Sriraman, 2010, pp. 224-225). It became tangible for the teachers that the environment is full of “treasures” (they used this word), affordances, let us say, that enable the user to express her/his mathematical ideas. There is no neutral artifact, no neutral environment. Each artifact *drives* the actions of the user (individual or collective) and is *driven* by the user in a coextensive process that leaves no one unchanged. As a cognitive agent, the user eventually incorporates the artifact to her cognitive resources. That is what we do when add two numbers: we do not *see,* anymore, the decimal notation system that after years of schooling has become incorporated as a cognitive instrument.

Among the treasures the teachers became aware of, sliders and dragging were instrumental for their mathematical thinking: These became instruments to deal with and control continuous and discrete variation. One of our goals was to help the teachers to develop conceptual and computational fluencies. That is, the tools affordances are a vehicle to

represent and explore concept meaning and its uses in problem solving activities. We believe this is possible if teachers have at their hands the mediation of dynamic, digital environments.
We feel we are in the middle of a transitional period along which, blending already built ideas in static media with their representational re-descriptions in terms of digital representations, will open windows into a new mathematical culture in the classroom.

The visual derivative. One of the inertial obstructions to develop a sound vision of derivative is the understanding of this concept as *a number*. Once the derivative has been introduced as the slope of the *tangent line to the graph,* at one point, this tangent is, in most cases used to locate maxima or minima of the function. This is done, following a mechanical procedure, almost a mantra: find the derivative; make it equal to zero...and so on. But the important step, find the derivative *and try to understand what it says about function globally,* is almost never taken. This is the inertial effect we talked about previously. It is part of school culture.
We decided to discuss this issue taking as a starting point the graph of a function and the tangent line at a point of the graph. Then we took a small segment of the tangent line around the point of tangency as shown in the next Figure. The idea we wanted to introduce was that a short, a very short indeed, segment of the tangent line around a point of tangency could generate the graph of the function. Next we dragged the segment (activating the trace for the segment) and we produced the figure:

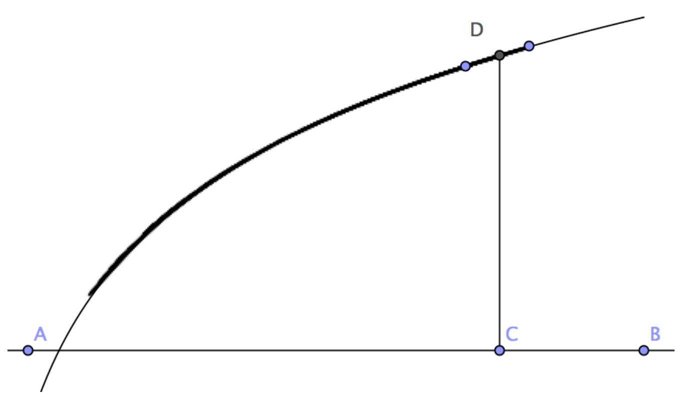

Figure 12: a tangent segment

Then a discussion began about the meaning of *close* when we say that the tangent line is the best approximation to the function around a point.

After a while we proposed the teachers to discuss the following situation: we hid the graph of the function but kept visible the tangent segment as in figure 13:

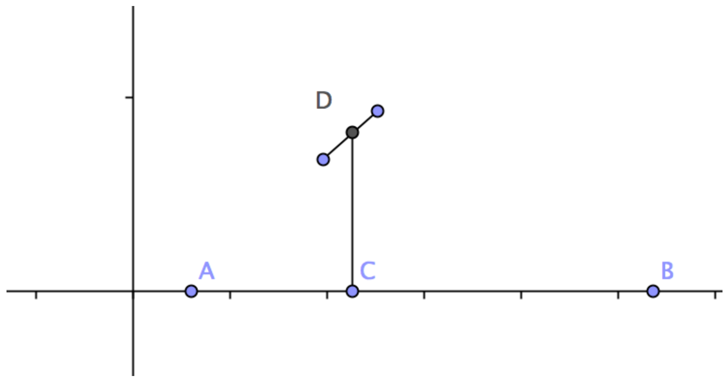

Figure 13: recovering the graph

Then, we dragged the segment (with the trace active) and showed that we could recover the curve, the whole curve.

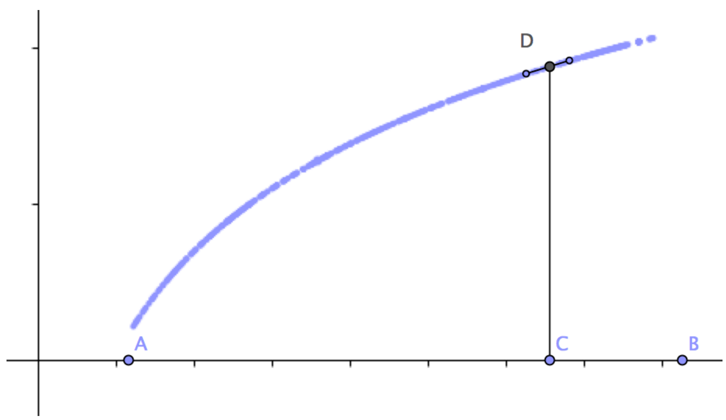

Figure 14

Some teachers were amazed and then one of them essentially asked: *What does the tangent know?* Another replied: *it does not have to know anything, because the curve is hidden but not erased*. So, in a sense dragging the segment was a way to *uncover* the hidden curve. We thought this discussion was very valuable indeed; something deep was floating in the atmosphere of the classroom. We thought time was right for making explicit, a seed about the fundamental theorem of Calculus. We have simply to extend some ideas that were already under discussion with the teachers. If you have the function then you have the tangent lines and reciprocally, if you have the tangent lines you can recover the function.

This time, our emphasis was in establishing that each time you have the function, in fact you have two functions: the one you already have and the derivative function that maps the behavior of the original function. In these tasks, we were trying to emphasize the conceptual fluency beyond the operational fluency that the teachers were more familiar with.

It has become clear from the vignettes and exemplars we have previously outlined, that our math intuitions relies in very specific ways on action and motion and the digital environment have provided our group of teachers a great service to transform some metaphorical thinking on motion and action, into sound cognitive instruments.

Final thoughts.

In 1996, world chess champion Garry Kasparov played a match against Deep Blue, an IBM supercomputer. Kasparov wrote in TIME magazine that he could feel, even *smell* a new kind of intelligence across the table. After almost 17 years, Kasparov's story seems up to date; this new intelligence shapes our actions and behaviors. The zeal that let our community to face the challenge of thinking about thinking has been moving indeed. This is an issue that has come to the fore because the presence of digital technologies has made unavoidable to consider the distribution of intelligence away from "its confining biological membrane" (Donald, 1991, p.359).

The *externalization of memory* inaugurated this momentous stage developed into symbolic technologies. But even if one could feel an intelligence sitting on the page of a novel, that was human intelligence indeed.

Kasparov's feelings had a different source. Societies are already (or will be sooner than later), saturated with the presence of visible and invisible computers. Not all of them play chess but some are able to control the flight of a huge airplane across the Pacific. Others can give us the location of the restaurant we are looking for, or compute a complex mathematical model that we, human beings, cannot compute with static symbolic technology alone.

If the power of digital technologies is broadly tangible, there is no reason to expect they will not have as well a profound impact at the level of formal education. Education has to cross that Rubicon and understand that the executable, symbolic, representations are key to make even more tangible the zone of potential development of social, distributed intelligence. Indeed, intelligence is a network phenomenon and we have to conceive of it globally, seamlessly, in a move that includes all kinds of intelligences, as those Kasparov caught a glimpse of across the table.

And what about schools? We might say that old habits die hard, but it is not just a matter of habits, it is more a matter of transformation of cultures.

The new classroom with the possibility of sharing an expressive medium, like the digital environment, can help us organize open mathematical discussions and foster a continuous reflection within a social space in permanent evolution. In this space, meaning of mathematical entities evolves with the opportunities to directly manipulate them. Mathematical entities, as explained previously, are only indirectly accessible through semiotic representations (Duval 2000) and consequently, the only way of gaining access to them is using, for instance, words, symbols, expressions or drawings. But no representation exhausts the represented entity. Nevertheless, any mathematical representation has such a crystallizing impact on how mathematical entities are experienced that when we work with it, we have the feeling of being working inside a Platonic mathematical reality. But this is only an illusion that lurks beneath the surface. Mathematical reality is a human reality even if it is a virtual one: one cannot forget humans have the power to extend their world of experience symbolically.

Closer to our professional interests is the *mode of existence* that teachers have experienced whilst working with dynamic geometry environment and when they analyzed and discussed a design activity whose goal was to construct a dynamic model involving a truck approaching an underpass. In all these activities, it is made tangible that we can explore and experiment on dynamic representations of mathematical entities as if they were material objects (Santos-Trigo & Reyes-Rodríguez, 2011). In fact, the executable nature of dynamic representations enables the learner to continuously modify those representations while preserving their structural features. This reflects a profound difference with the static representations of traditional mathematics at school. The kind of intelligence living in the executable representations extends human action with digital artifacts into the social space of the classroom. The end result of this process is an instrument loaded with the intelligence shared in the classroom. In practice, it took long weeks for the teachers to master and embody new ways of interacting with the virtual reality of digital

entities. No artifact is epistemologically neutral and consequently, there is a disruption in the taken for granted aspects of what it means to think mathematically in the digital contexts. In this view, an instrument, that is, the internalized artifact is a template for action. It is relevant here to mention that, with the instrument, the learner can explore new landscapes of mathematical validation. In fact, it comes to the fore the notion of *theorem in motion* embodied in the dynamic digital environment — This is how we conceive of it. Then, reconsidering the transformation of static entities through executable representations, we are opening a window to new mathematical entities, whose proper ecology is the digital. But it is not the search of the object per se what moves us as researchers, but the search for new ways of thinking.
We expect that the mathematics of change and variations, through their digital embodiments, contributes to a substantial gain in students' development of conceptual understanding and computational fluency.

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